

# On Fréchet differentiability of Lipschitz maps between Banach spaces

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## Abstract

A well-known open question is whether every countable collection of Lipschitz functions on a Banach space  $X$  with separable dual has a common point of Fréchet differentiability. We show that the answer is positive for some infinite-dimensional  $X$ . Previously, even for collections consisting of two functions this has been known for finite-dimensional  $X$  only (although for one function the answer is known to be affirmative in full generality). Our aims are achieved by introducing a new class of null sets in Banach spaces (called  $\Gamma$ -null sets), whose definition involves both the notions of category and measure, and showing that the required differentiability holds almost everywhere with respect to it. We even obtain existence of Fréchet derivatives of Lipschitz functions between certain infinite-dimensional Banach spaces; no such results have been known previously.

Our main result states that a Lipschitz map between separable Banach spaces is Fréchet differentiable  $\Gamma$ -almost everywhere provided that it is regularly Gâteaux differentiable  $\Gamma$ -almost everywhere and the Gâteaux derivatives stay within a norm separable space of operators. It is easy to see that Lipschitz maps of  $X$  to spaces with the Radon-Nikodým property are Gâteaux differentiable  $\Gamma$ -almost everywhere. Moreover, Gâteaux differentiability implies regular Gâteaux differentiability with exception of another kind of negligible sets, so-called  $\sigma$ -porous sets. The answer to the question is therefore positive in every space in which every  $\sigma$ -porous set is  $\Gamma$ -null. We show that this holds for  $C(K)$  with  $K$  countable compact, the Tsirelson space and for all subspaces of  $c_0$ , but that it fails for Hilbert spaces.

## 1. Introduction

One of the main aims of this paper is to show that infinite-dimensional Banach spaces may have the property that any countable collection of real-valued Lipschitz functions defined on them has a common point of Fréchet

differentiability. Previously, this has not been known even for collections consisting of two such functions. Our aims are achieved by introducing a new class of null sets in Banach spaces and proving results on differentiability almost everywhere with respect to it. The definition of these null sets involves both the notions of category and measure. This new concept even enables the proof of the existence of Fréchet derivatives of Lipschitz functions between certain infinite-dimensional Banach spaces. No such results have been known previously.

Before we describe this new class of null sets and the new results, we present briefly some background material (more details and additional references can be found in [3]).

There are two basic notions of differentiability for functions  $f$  defined on an open set in a Banach space  $X$  into a Banach space  $Y$ . The function  $f$  is said to be Gâteaux differentiable at  $x_0$  if there is a bounded linear operator  $T$  from  $X$  to  $Y$  so that for every  $u \in X$ ,

$$\lim_{t \rightarrow 0} \frac{f(x_0 + tu) - f(x_0)}{t} = Tu.$$

The operator  $T$  is called the Gâteaux derivative of  $f$  at  $x_0$  and is denoted by  $D_f(x_0)$ .

If for some fixed  $u$  the limit

$$f'(x_0, u) = \lim_{t \rightarrow 0} \frac{f(x_0 + tu) - f(x_0)}{t}$$

exists, we say that  $f$  has a directional derivative at  $x_0$  in the direction  $u$ . Thus  $f$  is Gâteaux differentiable at  $x_0$  if and only if all the directional derivatives  $f'(x_0, u)$  exist and they form a bounded linear operator of  $u$ . Note that in our notation we have in this case  $f'(x_0, u) = D_f(x_0)u$ .

If the limit in the definition of Gâteaux derivative exists uniformly in  $u$  on the unit sphere of  $X$ , we say that  $f$  is Fréchet differentiable at  $x_0$  and  $T$  is the Fréchet derivative of  $f$  at  $x_0$ . Equivalently,  $f$  is Fréchet differentiable at  $x_0$  if there is a bounded linear operator  $T$  such that

$$f(x_0 + u) = f(x_0) + Tu + o(\|u\|) \text{ as } \|u\| \rightarrow 0.$$

It is trivial that if  $f$  is Lipschitz and  $\dim(X) < \infty$  then the notion of Gâteaux differentiability and Fréchet differentiability coincide. The situation is known to be completely different if  $\dim(X) = \infty$ . In this case there are reasonably satisfactory results on the existence of Gâteaux derivatives of Lipschitz functions, while results on existence of Fréchet derivatives are rare and usually very hard to prove. On the other hand, in many applications it is important to have Fréchet derivatives of  $f$ , since they provide genuine local linear approximation to  $f$ , unlike the much weaker Gâteaux derivatives.

Before we proceed we mention that we shall always assume the domain space to be separable and therefore also  $Y$  can be assumed to be separable.

We state now the main existence theorem for Gâteaux derivatives. This is a direct and quite simple generalization of Rademacher's theorem to infinite-dimensional spaces. But first we recall the definition of two notions which enter into its statement.

A Banach space  $Y$  is said to *have the Radon-Nikodým property* (RNP) if every Lipschitz function  $f : \mathbb{R} \rightarrow Y$  is differentiable almost everywhere (or equivalently every such  $f$  has a point of differentiability).

A Borel set  $A$  in  $X$  is said to be *Gauss null* if  $\mu(A) = 0$  for every nondegenerate (i.e. not supported on a proper closed hyperplane) Gaussian measure  $\mu$  on  $X$ . There is also a related notion of Haar null sets which will not be used in this paper. We just mention, for the sake of orientation, that the class of Gauss null sets forms a proper subset of the class of Haar null sets.

**THEOREM 1.1** ([4], [9], [1]). *Let  $X$  be separable and  $Y$  have the RNP. Then every Lipschitz function from an open set  $G$  in  $X$  into  $Y$  is Gâteaux differentiable outside a Gauss null set.*

In view of the definition of the RNP, the assumption on  $Y$  in Theorem 1.1 is necessary. Easy and well-known examples show that Theorem 1.1 fails badly if we want Fréchet derivatives. For example, the map  $f : \ell_2 \rightarrow \ell_2$  defined by  $f(x_1, x_2, \dots) = (|x_1|, |x_2|, \dots)$  is nowhere Fréchet differentiable.

In the study of Fréchet differentiability there is another notion of smallness of sets which enters naturally in many contexts. A set  $A$  in a Banach space  $X$  (and even in a general metric space) is called *porous* if there is a  $0 < c < 1$  so that for every  $x \in A$  there are  $\{y_n\}_{n=1}^\infty \subset X$  with  $y_n \rightarrow x$  and so that  $B(y_n, c \operatorname{dist}(y_n, x)) \cap A = \emptyset$  for every  $n$ . (We denote by  $B(z, r)$  the closed ball with center  $z$  and radius  $r$ .) An important reason for the connections between porous sets and Fréchet differentiability is the trivial remark that if  $A$  is porous in a Banach space  $X$  then the Lipschitz function  $f(x) = \operatorname{dist}(x, A)$  is not Fréchet differentiable at any point of  $A$ . Indeed, the only possible value for the (even Gâteaux) derivative of  $f$  at  $x \in A$  is zero. But with  $y_n$  and  $c$  as above,  $f(y_n) \geq c \operatorname{dist}(y_n, x)$  is not  $o(\operatorname{dist}(y_n, x))$  as  $n \rightarrow \infty$ . A set  $A$  is called  *$\sigma$ -porous* if it can be represented as a union  $A = \bigcup_{n=1}^\infty A_n$  of countably many porous sets (the porosity constant  $c_n$  may vary with  $n$ ).

If  $U$  is a subspace of a Banach space  $X$ , then the set  $A$  will be called *porous in the direction  $U$*  if there is a  $0 < c < 1$  so that for every  $x \in A$  and  $\varepsilon > 0$  there is a  $u \in U$  with  $\|u\| < \varepsilon$  and so that  $B(x + u, c\|u\|) \cap A = \emptyset$ . A set  $A$  in a Banach space  $X$  is called *directionally porous* if there is a  $0 < c < 1$  so that for every  $x \in A$  there is a  $u = u(x)$  with  $\|u\| = 1$  and a sequence  $\lambda_n \searrow 0$  so that  $B(x + \lambda_n u, c\lambda_n) \cap A = \emptyset$  for every  $n$ . The notions of  *$\sigma$ -porous sets in the direction  $U$*  or  *$\sigma$ -directionally porous sets* are defined in an obvious way.

In finite-dimensional spaces a simple compactness argument shows that the notions of porous and directionally porous sets coincide. As it will become presently clear, this is not the case if  $\dim(X) = \infty$ . In finite-dimensional spaces porous sets are small from the point of view of measure (they are of Lebesgue measure zero by Lebesgue's density theorem) as well as category (they are obviously of the first category). In infinite-dimensional spaces only the first category statement remains valid.

As is well known, the easiest class of functions to handle in differentiation theory are convex continuous real-valued functions  $f : X \rightarrow \mathbb{R}$ . In [14] it is proved that if  $X^*$  is separable then any convex continuous  $f : X \rightarrow \mathbb{R}$  is Fréchet differentiable outside a  $\sigma$ -porous set. In separable spaces with  $X^*$  nonseparable it is known [7] that there are convex continuous functions (even equivalent norms) which are nowhere Fréchet differentiable. It is shown in [10] and [11] that in every infinite-dimensional super-reflexive space  $X$ , and in particular in  $\ell_2$ , there is an equivalent norm which is Fréchet differentiable only on a Gauss null set. It follows that such spaces  $X$  can be decomposed into the union of two Borel sets  $A \cup B$  with  $A$   $\sigma$ -porous and  $B$  Gauss null. Such a decomposition was proved earlier and directly for every separable infinite-dimensional Banach space  $X$  (see [13]). Note that if  $A$  is a directionally porous set in a Banach space then, by an argument used already above, the Lipschitz function  $f(x) = \text{dist}(x, A)$  is not even Gâteaux differentiable at any point  $x \in A$  and thus by Theorem 1.1 the set  $A$  is Gauss null.

The new null sets (called  $\Gamma$ -null sets) will be introduced in the next section. There we prove some simple facts concerning these null sets and in particular that Theorem 1.1 also holds if we require the exceptional set (i.e. the set of non-Gâteaux differentiability) to be  $\Gamma$ -null.

The main result on Fréchet differentiability in the context of  $\Gamma$ -null sets is proved in Section 3. From this result it follows in particular that if every  $\sigma$ -porous set in  $X$  is  $\Gamma$ -null then any Lipschitz  $f : X \rightarrow Y$  with  $Y$  having the RNP whose set of Gâteaux derivatives  $\{D_f(x)\}$  is separable is Fréchet differentiable  $\Gamma$ -almost everywhere. From the main result it follows also that convex continuous functions on any space  $X$  with  $X^*$  separable are Fréchet differentiable  $\Gamma$ -almost everywhere. In particular, if  $X^*$  is separable,  $f : X \rightarrow \mathbb{R}$  is convex and continuous and  $g : X \rightarrow Y$  is Lipschitz with  $Y$  having the RNP then there is a point  $x$  (actually  $\Gamma$ -almost any point) at which  $f$  is Fréchet differentiable and  $g$  is Gâteaux differentiable. This information on existence of such an  $x$  cannot be deduced from the previously known results. It is also clear from what was said above that the  $\Gamma$ -null sets and Gauss null sets form completely different  $\sigma$ -ideals in general (the space  $X$  can be decomposed into disjoint Borel sets  $A_0 \cup B_0$  with  $A_0$  Gauss null and  $B_0$   $\Gamma$ -null, at least when  $X$  is infinite-dimensional and super-reflexive).

In Section 4 we prove that for  $X = c_0$  or more generally  $X = C(K)$  with  $K$  countable compact and for some closely related spaces that every  $\sigma$ -porous set in them is indeed  $\Gamma$ -null. Thus combined with the main result of Section 3 we get a general result on existence of points of Fréchet differentiability for Lipschitz maps  $f : X \rightarrow Y$  where  $X$  is as above and  $Y$  has the RNP. This is the first result on existence of points of Fréchet differentiability for general Lipschitz mappings for certain pairs of infinite-dimensional spaces. Actually, the only previously known general result on existence of points of Fréchet differentiability of Lipschitz maps with infinite-dimensional domain dealt with maps whose range is the real line [12] and [8].

Unfortunately, the class of spaces in which  $\sigma$ -porous sets are  $\Gamma$ -null does not include the Hilbert space  $\ell_2$  or more generally  $\ell_p$ ,  $1 < p < \infty$ . The reason for this is an example in [13] which shows that for these spaces the mean value theorem for Fréchet derivatives fails while a result in Section 5 shows that in the sense of  $\Gamma$ -almost everywhere the mean value theorem for Fréchet derivatives holds. All this is explained in detail in Section 5.

The paper concludes in Section 6 with some comments and open problems.

## 2. $\Gamma$ -null sets

Let  $T = [0, 1]^{\mathbb{N}}$  be endowed with the product topology and product Lebesgue measure  $\mu$ . Let  $\Gamma(X)$  be the space of continuous mappings  $\gamma : T \rightarrow X$  having continuous partial derivatives  $D_j\gamma$  (with one-sided derivatives at points where the  $j$ -th coordinate is 0 or 1). The elements of  $\Gamma(X)$  will be called surfaces. For finitely supported  $s \in \ell_\infty$  we also use the notation  $\gamma'(t)(s) = \sum_{j=1}^{\infty} s_j D_j\gamma(t)$ . We equip  $\Gamma(X)$  by the topology generated by the semi-norms  $\|\gamma\|_0 = \sup_{t \in T} \|\gamma(t)\|$  and  $\|\gamma\|_k = \sup_{t \in T} \|D_k\gamma(t)\|$ . Equivalently, this topology may be defined by using the semi-norms  $\|\gamma\|_{\leq k} = \max_{0 \leq j \leq k} \|\gamma\|_j$ . The space  $\Gamma(X)$  with this topology is a Fréchet space; in particular, it is Polish (metrizable by a complete separable metric).

We will often use the simple observation that for every  $\gamma \in \Gamma(X)$ ,  $m \in \mathbb{N}$  and  $\varepsilon > 0$  there is  $n \in \mathbb{N}$  so that for every  $t \in T$  the surface

$$\gamma^{n,t}(s) = \gamma(s_1, \dots, s_n, t_{n+1}, t_{n+2}, \dots)$$

satisfies

$$\|\gamma^{n,t} - \gamma\|_{\leq m} < \varepsilon.$$

This follows immediately from the uniform continuity of  $\gamma$  and its partial derivatives. We let  $\Gamma_k(X)$  be the space of those  $\gamma \in \Gamma(X)$  that depend on the first  $k$  coordinates of  $T$  and note that by the observation above  $\bigcup_{k=1}^{\infty} \Gamma_k(X)$  is dense in  $\Gamma(X)$ .

The *tangent space*  $\text{Tan}(\gamma, t)$  of  $\gamma$  at a point  $t \in T$  is defined to be the closed linear span in  $X$  of the vectors  $\{D_k \gamma(t)\}_{k=1}^\infty$ .

*Definition 2.1.* A Borel set  $N \subset X$  will be called  $\Gamma$ -null if  $\mu\{t \in T : \gamma(t) \in N\} = 0$  for residually many  $\gamma \in \Gamma(X)$ . A possibly non-Borel set  $A \subset X$  will be called  $\Gamma$ -null if it is contained in a Borel  $\Gamma$ -null set.

Sometimes, we will also consider  $T$  as a subset of  $\ell_\infty$ . For example, for  $s, t \in T$  we use the notation  $\|s - t\| = \sup_{j \in \mathbb{N}} |s_j - t_j|$ . We also use the notation  $Q_k(t, r) = \{s \in T : \max_{1 \leq j \leq k} |s_j - t_j| \leq r\}$ .

**LEMMA 2.2.** *Let  $\{u_j\}_{j=1}^n \subset X$  and  $\varepsilon > 0$ . Then the set of those  $\gamma \in \Gamma(X)$  for which there are  $k \in \mathbb{N}$  and  $c > 0$  such that*

$$\max_{1 \leq j \leq n} \sup_{t \in T} \|cD_{k+j}\gamma(t) - u_j\| < \varepsilon$$

*is dense and open in  $\Gamma(X)$ .*

*Proof.* By the definition of the topology of  $\Gamma(X)$  it is clear that this set is open. To see that it is dense it suffices to show that its closure contains  $\Gamma_k(X)$  for every  $k$ . Let  $\gamma_0 \in \Gamma_k(X)$ ,  $\eta > 0$  and consider the surface  $\gamma(t) = \gamma_0(t) + \eta \sum_{j=1}^n t_{k+j} u_j$ . Then  $\|\gamma - \gamma_0\|_0 \leq n\eta \max_{1 \leq j \leq n} \|u_j\|$ ,  $\|\gamma - \gamma_0\|_l = 0$  if  $l \leq k$  or  $l > k+n$ ,  $\|\gamma - \gamma_0\|_{k+j} = \eta \|u_j\|$  and  $cD_{k+j}\gamma(t) = u_j$  for  $1 \leq j \leq k$  and  $c = 1/\eta$ .  $\square$

**COROLLARY 2.3.** *If  $X$  is separable, then residually many  $\gamma \in \Gamma(X)$  have the property that  $\text{Tan}(\gamma, t) = X$  for every  $t \in T$ .*

*Proof.* By Lemma 2.2 (with  $n = 1$ ) we get that for every  $u \in X$  the set of those  $\gamma \in \Gamma(X)$  such that  $\text{dist}(u, \text{Tan}(\gamma, t)) < \varepsilon$  for every  $t \in T$  is open and dense in  $\Gamma(X)$ . The desired result follows now from the separability of  $X$ .  $\square$

We show next that the class of  $\Gamma$ -null sets in a finite-dimensional space  $X$  coincides with the class of sets of Lebesgue measure zero (just as for Gauss and Haar null sets).

**THEOREM 2.4.** *In finite-dimensional spaces,  $\Gamma$ -null sets coincide with Lebesgue null sets.*

*Proof.* Let  $u_1, \dots, u_n \in X$  be a basis for  $X$ , let  $E \subset X$  be a Borel set and denote by  $|E|$  its Lebesgue measure.

If  $|E| > 0$ , define  $\gamma_0 : T \rightarrow X$  by  $\gamma_0(t) = u + \sum_{j=1}^n t_j u_j$ , where  $u \in X$  is chosen so that  $|E \cap \gamma_0(T)| > 0$ . If  $\|\gamma - \gamma_0\|_{\leq n}$  is sufficiently small, then for every  $s = (s_1, s_2, \dots) \in T$  the mappings  $\gamma_s(t_1, \dots, t_n) = \gamma(t_1, \dots, t_n, s_1, s_2, \dots)$  are

diffeomorphisms of  $[0, 1]^n$  onto subsets of  $X$  which meet  $E$  in a set of measure at least  $|E \cap \gamma_0(T)|/2$ . Hence, for every  $s$ ,  $|\gamma_s^{-1}(E)| \geq c_1|E \cap \gamma_0(T)|$ , for a suitable positive constant  $c_1$ . Hence

$$\mu(\gamma^{-1}(E)) = \int_T |\gamma_s^{-1}(E)| d\mu(s) \geq c_1|E \cap \gamma_0(T)|$$

and we infer that  $E$  is not  $\Gamma$ -null.

If  $|E| = 0$ , we use Lemma 2.2 with a sufficiently small  $\varepsilon > 0$  to find a dense open set of such surfaces  $\gamma \in \Gamma(X)$  for which there are  $k \in \mathbb{N}$  and  $c > 0$  such that

$$\max_{1 \leq j \leq n} \sup_{t \in T} \|D_{k+j} c\gamma(t) - u_j\| < \varepsilon.$$

Then the mappings  $c\gamma_s(t_1, \dots, t_n) = c\gamma(s_1, \dots, s_k, t_1, \dots, t_n, s_{k+1}, s_{k+2}, \dots)$  are, for every  $s \in T$ , diffeomorphisms of  $[0, 1]^n$  onto a subset of  $X$ . The same is therefore true for the mappings  $\gamma_s$ ,  $s \in T$ . Hence  $|\gamma_s^{-1}(E)| = 0$  for every  $s$  and hence

$$\mu(\gamma^{-1}(E)) = \int_T |\gamma_s^{-1}(E)| d\mu(s) = 0;$$

i.e.  $E$  is  $\Gamma$ -null. □

We show next that Theorem 1.1 remains valid if we replace in its statement Gauss null sets by  $\Gamma$ -null sets.

**THEOREM 2.5.** *Let  $X$  be separable and  $Y$  have the RNP. Then every Lipschitz function from an open set  $G$  in  $X$  into  $Y$  is Gâteaux differentiable outside a  $\Gamma$ -null set.*

*Proof.* We remark first that the set of points at which  $f$  fails to be Gâteaux differentiable is a Borel set. We recall next that Rademacher's theorem holds also for Lipschitz maps from  $\mathbb{R}^k$  to a space  $Y$  having the RNP (see e.g. [3, Prop. 6.41]). Consider now an arbitrary surface  $\gamma$ . By using Fubini's theorem, we get that for almost every  $t \in T$  for which  $\gamma(t) \in G$  the mapping

$$(s_1, \dots, s_k) \rightarrow f(\gamma(s_1, \dots, s_k, t_{k+1}, \dots))$$

is differentiable at  $s = (t_1, \dots, t_k)$ . Since  $f$  is Lipschitz, it follows that, for almost all  $t$ ,  $f$  has directional derivatives for all vectors

$$v \in \overline{\text{span}}\{D_j(\gamma)(t)\}_{j=1}^\infty = \text{Tan}(\gamma, t)$$

at  $u = \gamma(t)$  and that these directional derivatives depend linearly on  $v$  (see e.g. [3, Lemma 6.40]). In particular, for every surface  $\gamma$  from the residual set obtained in Corollary 2.3  $f$  is Gâteaux differentiable at  $u = \gamma(t)$  for almost all  $t \in T$  for which  $\gamma(t) \in G$ . This proves the theorem. □

### 3. Fréchet differentiability

In this section we prove the main criterion for Fréchet differentiability of Lipschitz functions in terms of  $\Gamma$ -null sets. But first we have to introduce the following simple notion.

*Definition 3.1.* Suppose that  $f$  is a map from (an open set in)  $X$  to  $Y$ . We say that a point  $x$  is a *regular point of  $f$*  if for every  $v \in X$  for which  $f'(x, v)$  exists,

$$\lim_{t \rightarrow 0} \frac{f(x + tu + tv) - f(x + tu)}{t} = f'(x, v)$$

uniformly for  $\|u\| \leq 1$ .

Note that in the definition above it is enough to take the limit for  $t \searrow 0$  only, since we may replace  $v$  by  $-v$ .

**PROPOSITION 3.2.** *For a convex continuous function  $f : X \rightarrow \mathbb{R}$  every point  $x$  is a regular point of  $f$ .*

*Proof.* Given  $x \in X$ ,  $v \in X$  and  $\varepsilon > 0$ , find  $r > 0$  such that

$$|(f(x + tv) - f(x))/t - f'(x, v)| < \varepsilon$$

for  $0 < |t| < r$  and such that  $f$  is Lipschitz on  $B(x, 2r(1 + \|v\|))$  with constant  $K$ . If  $\|u\| \leq 1$  and  $0 < t < \min(r, \varepsilon r/2K)$ , then

$$\begin{aligned} (f(x + tu + tv) - f(x + tu))/t &\leq (f(x + tu + rv) - f(x + tu))/r \\ &\leq (f(x + rv) - f(x))/r + 2Kt\|u\|/r \\ &< f'(x, v) + 2\varepsilon \end{aligned}$$

and

$$\begin{aligned} (f(x + tu + tv) - f(x + tu))/t &\geq (f(x + tu) - f(x + tu - rv))/r \\ &\geq (f(x) - f(x - rv))/r - 2Kt\|u\|/r \\ &> f'(x, v) - 2\varepsilon. \end{aligned} \quad \square$$

*Remark.* It is well known and as easy to prove as the statement above that convex functions satisfy a stronger condition of regularity (sometimes called Clarke regularity), namely that

$$\lim_{z \rightarrow x, t \rightarrow 0} \frac{f(z + tv) - f(z)}{t} = f'(x, v)$$

whenever  $f'(x, v)$  exists. We do not use here this stronger regularity concept since while every point of Fréchet differentiability of  $f$  is a point of regularity of  $f$  in our sense, this no longer holds for the stronger regularity notion; this is immediate by considering an indefinite integral of the characteristic function



of a set  $E \subset \mathbb{R}$  such that both  $E$  and its complement have positive measure in every interval. Therefore the stronger form of regularity cannot be used in proving existence of points of differentiability for Lipschitz maps (which is our purpose here).

**PROPOSITION 3.3.** *Let  $f$  be a Lipschitz map from an open subset  $G$  of a separable Banach space  $X$  to a separable Banach space  $Y$ . Then the set of irregular points of  $f$  is  $\sigma$ -porous.*

*Proof.* For  $p, q \in \mathbb{N}$ ,  $v$  from a countable dense subset of  $X$  and  $w$  from a countable dense subset of  $Y$  let  $E_{p,q,v,w}$  be the set of those  $x \in X$  such that  $\|f(x + tv) - f(x) - tw\| \leq |t|/p$  for  $|t| < 1/q$ , and

$$\limsup_{t \rightarrow 0} \sup_{\|u\| \leq 1} \left\| \frac{f(x + tu + tv) - f(x + tu)}{t} - w \right\| > 2/p.$$

Whenever  $x \in E_{p,q,v,w}$ , there are arbitrarily small  $|t| < 1/q$  such that for some  $u$  with  $\|u\| \leq 1$ ,

$$\|f(x + tu + tv) - f(x + tu) - tw\| > 2|t|/p.$$

If  $\|y - (x + tu)\| < |t|/2p\text{Lip}(f)$ , then

$$\|f(y + tv) - f(y) - tw\| \geq \|f(x + tu + tv) - f(x + tu) - tw\| - |t|/p > |t|/p$$

and hence  $y \notin E_{p,q,v,w}$ . This proves that  $E_{p,q,v,w}$  is  $1/2p\text{Lip}(f)$  porous. Since every irregular point of  $f$  belongs to some  $E_{p,q,v,w}$  the result follows.  $\square$

The next lemma is a direct consequence of the definition of regularity. It will make the use of the regularity assumption more convenient in subsequent arguments.

**LEMMA 3.4.** *Suppose that  $f$  is Lipschitz on a neighborhood of  $x$  and that, at  $x$ , it is regular and differentiable in the direction of a finite-dimensional subspace  $V$  of  $X$ . Then for every  $C$  and  $\varepsilon > 0$  there is a  $\delta > 0$  such that*

$$\|f(x + v + u) - f(x + v)\| \geq \|f(x + u) - f(x)\| - \varepsilon\|u\|$$

whenever  $\|u\| \leq \delta$ ,  $v \in V$  and  $\|v\| \leq C\|u\|$ .

*Proof.* Let  $r > 0$  be such that  $f$  is Lipschitz on  $B(x, r)$ . Let  $S$  be a finite subset of  $\{v \in V : \|v\| \leq C\}$  such that for every  $w$  in this set there is  $v \in S$  such that  $\|w - v\| < \varepsilon/6\text{Lip}(f)$ . By the definition of regularity, there is  $0 < \delta < r/(1 + C)$  such that

$$(1) \quad \|f(x + t\hat{u} + t\hat{v}) - f(x + t\hat{u}) - tf'(x, \hat{v})\| \leq \varepsilon t/3$$

whenever  $0 \leq t \leq \delta$ ,  $\|\hat{u}\| \leq 1$  and  $\hat{v} \in S$ .

Suppose that  $\|u\| \leq \delta$ ,  $v \in V$  and  $\|v\| \leq C\|u\|$ . By using (1) with  $t = \|u\|$ ,  $\hat{u} = u/\|u\|$  and  $\hat{v} \in S$  with  $\|\hat{v} - v/\|u\|\| < \varepsilon/6\text{Lip}(f)$ , we get

$$(2) \quad \|f(x + u + t\hat{v}) - f(x + u) - tf'(x, \hat{v})\| \leq \varepsilon t/3.$$

Similarly by using (1) with the same  $t$  and  $\hat{v}$  but with  $\hat{u} = 0$ , we get

$$(3) \quad \|f(x + t\hat{v}) - f(x) - tf'(x, \hat{v})\| \leq \varepsilon t/3.$$

Hence by the triangle inequality, (2) and (3) we deduce that

$$\|(f(x + t\hat{v} + u) - f(x + t\hat{v})) - (f(x + u) - f(x))\| \leq 2\varepsilon t/3.$$

Recalling that  $t = \|u\|$  we thus get

$$\begin{aligned} & \|f(x + v + u) - f(x + v)\| \\ & \geq \|f(x + t\hat{v} + u) - f(x + t\hat{v})\| - 2\text{Lip}(f)\|v - t\hat{v}\| \\ & \geq \|f(x + u) - f(x)\| - 2\varepsilon t/3 - 2\text{Lip}(f)t(\varepsilon/6\text{Lip}(f)) \\ & = \|f(x + u) - f(x)\| - \varepsilon\|u\|. \end{aligned} \quad \square$$

Our next lemma examines the consequence of having a Lipschitz function which is not Fréchet differentiable at a regular point: It shows that if a surface contains such a point  $x$  then, after a suitable small perturbation of the surface, the derivative near  $x$ , in the mean, is not close to its value at  $x$ . Moreover, this property persists for all surfaces close enough to the perturbed surface. As in the proof of Lemma 2.2 (and other proofs in §2) we will make here an essential use of the fact that in the context of  $\Gamma$ -null sets we work with infinite-dimensional surfaces  $\gamma \in \Gamma(X)$ . These surfaces can be well approximated by  $k$ -dimensional surfaces in  $\Gamma_k(X)$ . The surfaces in  $\Gamma_k(X)$  can in turn be approximated by surfaces in  $\Gamma_{k+1}(X)$  and we are quite free to do appropriate constructions on the  $(k+1)$ -th coordinate in order to get an approximation with desired properties.

**LEMMA 3.5.** *Let  $f : G \rightarrow Y$  be a Lipschitz function with  $G$  an open set in a separable Banach space  $X$ . Let  $E$  be a Borel subset of  $G$  consisting of points where  $f$  is Gâteaux differentiable and regular. Let  $\eta > 0$ ,  $\tilde{\gamma} \in \Gamma_k(X)$ ,  $t \in T$  so that  $x = \tilde{\gamma}(t) \in E$ .*

*Then there are  $0 < r < \eta$ ,  $\delta > 0$  and  $\hat{\gamma} \in \Gamma_{k+1}(X)$  such that  $\|\hat{\gamma} - \tilde{\gamma}\|_{\leq k} < \eta$ ,  $\hat{\gamma}(s) = \tilde{\gamma}(s)$  for  $s \in T \setminus Q_k(t, r)$  and so that the following holds: Whenever  $\gamma \in \Gamma(X)$  has the property that*

$$\|\gamma(s) - \hat{\gamma}(s)\| + \|D_{k+1}(\gamma(s) - \hat{\gamma}(s))\| < \delta, \quad s \in Q_k(t, r)$$

*then either*

$$\mu(Q_k(t, r) \setminus \gamma^{-1}(E)) \geq \alpha\mu(Q_k(t, r))/8\text{Lip}(f)$$

or

$$\int_{Q_k(t,r) \cap \gamma^{-1}(E)} \|D_f(\gamma(s)) - D_f(x)\| d\mu(s) \geq \alpha \mu(Q_k(t,r))/2,$$

where  $\alpha = \limsup_{u \rightarrow 0} \|f(x+u) - f(x) - D_f(x)u\|/\|u\|$ .

*Proof.* We shall assume that  $\alpha > \eta > 0$  and will define  $r$  and  $\delta$  later in the proof. Let  $0 < \zeta < \min(1, \alpha/5)$  be such that

$$(\alpha - 5\zeta)\mu(Q_k(t, (1 - \zeta)r)) \geq 3\alpha\mu(Q_k(t, r))/4$$

whenever  $t \in T$  and  $r < 1$ . This can evidently be accomplished even though for some  $t$ ,  $Q_k(t, r)$  is not an entire cube.

Choose a continuously differentiable function  $\omega : \ell_\infty \rightarrow [0, 1]$  depending on the first  $k$  coordinates only such that  $\omega(s) = 1$  if  $\|\pi_k s\| \leq 1 - \zeta$  and  $\omega(s) = 0$  if  $\|\pi_k s\| \geq 1$  where  $\pi_k$  denotes the projection of  $\ell_\infty$  on its first  $k$  coordinates. Let  $\max(4, \alpha) \leq K < \infty$  be such that  $\|\tilde{\gamma}(s_1) - \tilde{\gamma}(s_2)\| \leq K\|s_1 - s_2\|$ ,  $\|\omega(s_1) - \omega(s_2)\| \leq K\|s_1 - s_2\|$ , and the function

$$g(z) = f(z) - f(x) - D_f(x)(z - x)$$

is Lipschitz with constant  $K$  on  $B(x, 2(1 + K^2/\eta)\delta_1) \subset G$  for some  $0 < \delta_1 < \eta$ .

We let  $C = 4K^2/\eta$  and use Lemma 3.4 to find a  $0 < \delta_2 < \delta_1 < \eta$  such that

$$(4) \quad \|g(x+v+u) - g(x+v)\| \geq \|g(x+u)\| - \zeta\|u\|$$

whenever  $\|u\| \leq \delta_2$ ,  $v \in \text{Tan}(\tilde{\gamma}, t)$  (which is of dimension at most  $k$  since  $\tilde{\gamma} \in \Gamma_k(X)$ ) and  $\|v\| \leq C\|u\|$ . Also, let  $\delta_3 > 0$  be such that

$$(5) \quad \|\tilde{\gamma}(s) - x - \tilde{\gamma}'(t)(s - t)\| \leq \zeta\|s - t\|/C$$

whenever  $s \in T$  and  $\|\pi_k(s - t)\| \leq \delta_3$ . By the definition of  $\alpha$  we can find a  $u \in X$  such that

$$0 < \|u\| < \min(\delta_2, 2K\delta_3/C, \eta^2/2K)$$

and  $\|g(x+u)\| \geq (\alpha - \zeta)\|u\|$ .

Whenever  $s \in T$  and  $\|\pi_k(s - t)\| < C\|u\|/2K$ , we use (5) (which is applicable since  $C\|u\|/2K < \delta_3$ ) and (4) with  $v = \tilde{\gamma}'(t)(s - t)$  (this is justified by  $\|\tilde{\gamma}'(t)(s - t)\| \leq K\|\pi_k(s - t)\| < C\|u\|$ ) to infer that

$$\begin{aligned} (6) \quad & \|g(\tilde{\gamma}(s) + u) - g(\tilde{\gamma}(s))\| \\ & \geq \|g(x+v+u) - g(x+v)\| - 2K\|\tilde{\gamma}(s) - (x+v)\| \\ & \geq \|g(x+v+u) - g(x+v)\| - 2K\zeta\|s - t\|/C \\ & \geq \|g(x+u)\| - \zeta\|u\| - \zeta\|u\| \\ & \geq (\alpha - 3\zeta)\|u\|. \end{aligned}$$

We now define  $r$  to be  $2K\|u\|/\eta$  and put  $\hat{\gamma}(s) = \tilde{\gamma}(s) + s_{k+1}\omega((s-t)/r)u$ . By the choice of  $\|u\|$  we have  $0 < r < \eta$ . Also,  $\|\hat{\gamma} - \tilde{\gamma}\| \leq \|u\| < \eta$  and  $\|D_j(\hat{\gamma} - \tilde{\gamma})\| \leq K\|u\|/r < \eta$  for  $1 \leq j \leq k$  and thus  $\|\hat{\gamma} - \tilde{\gamma}\|_{\leq k} < \eta$  as required.

We show that the statement of the lemma holds with  $\delta = \zeta\|u\|/2K$ . Suppose that

$$\|\gamma(s) - \hat{\gamma}(s)\| + \|D_{k+1}(\gamma(s) - \hat{\gamma}(s))\| < \delta \text{ for } s \in Q_k(t, r)$$

and that

$$\mu(Q_k(t, r) \setminus \gamma^{-1}(E)) < \alpha\mu(Q_k(t, r))/8\text{Lip}(f).$$

Consider any  $s \in Q_k(t, (1 - \zeta)r)$  and put

$$\bar{s} = s + (1 - s_{k+1})e_{k+1} \text{ and } \underline{s} = s - s_{k+1}e_{k+1}$$

where  $e_{k+1}$  denotes the  $(k+1)$ -th unit vector in  $\ell_\infty$ ,  $I_s = [-s_{k+1}, 1 - s_{k+1}]$  and  $J_s = \{\sigma \in I_s : \gamma(s + \sigma e_{k+1}) \in E\}$ . Let  $u^* \in Y^*$  be such that  $\|u^*\| = 1$  and  $u^*(g(\gamma(\bar{s})) - g(\gamma(\underline{s}))) = \|g(\gamma(\bar{s})) - g(\gamma(\underline{s}))\|$ . Then  $\|\pi_k(s - t)\| < r = 2K\|u\|/\eta = C\|u\|/2K$  and by using (6) and the inequalities  $\|\gamma(\bar{s}) - \hat{\gamma}(\bar{s})\|, \|\gamma(\underline{s}) - \hat{\gamma}(\underline{s})\| < \delta$  and  $\text{Lip}(g) \leq K$ , we get

$$\begin{aligned} \int_{I_s} \frac{\partial u^*(g(\gamma(s + \sigma e_{k+1})))}{\partial \sigma} d\sigma &= \|g(\gamma(\bar{s})) - g(\gamma(\underline{s}))\| \\ &\geq \|g(\hat{\gamma}(\bar{s})) - g(\hat{\gamma}(\underline{s}))\| - \zeta\|u\| \\ &= \|g(\tilde{\gamma}(s) + u) - g(\tilde{\gamma}(s))\| - \zeta\|u\| \\ &\geq (\alpha - 4\zeta)\|u\|. \end{aligned}$$

Note that since  $s \in Q_k(t, (1 - \zeta)r)$  we have  $\frac{\partial \hat{\gamma}(s + \sigma e_{k+1})}{\partial \sigma} = u$  and

$$\|D_{k+1}(\gamma(s + \sigma e_{k+1}) - \hat{\gamma}(s + \sigma e_{k+1}))\| < \delta$$

and hence  $\|\frac{\partial \gamma(s + \sigma e_{k+1})}{\partial \sigma} - u\| < \delta$  for all  $\sigma \in I_s$ . Using also that  $\text{Lip}(g) \leq 2\text{Lip}(f)$  and that  $K \geq \max(4, \alpha)$ , we infer that

$$\begin{aligned} &\int_{J_s} \|D_g(\gamma(s + \sigma e_{k+1}))\| d\sigma \\ &\geq \int_{I_s} \left| \frac{\partial u^*(g(\gamma(s + \sigma e_{k+1})))}{\partial \sigma} \right| d\sigma / (\|u\| + \delta) - |I_s \setminus J_s| \text{Lip}(g) \\ &\geq (\alpha - 4\zeta)/(1 + \zeta/2K) - 2|I_s \setminus J_s| \text{Lip}(f) \\ &\geq (\alpha - 4\zeta)(1 - \zeta/2K) - 2|I_s \setminus J_s| \text{Lip}(f) \\ &\geq \alpha - 5\zeta - 2|I_s \setminus J_s| \text{Lip}(f). \end{aligned}$$

Integrating over  $s$  we obtain the desired result:

$$\begin{aligned}
& \int_{Q_k(t,r) \cap \gamma^{-1}(E)} \|D_g(\gamma(s))\| d\mu(s) \\
& \geq \int_{Q_k(t,(1-\zeta)r) \cap \gamma^{-1}(E)} \|D_g(\gamma(s))\| d\mu(s) \\
& \geq (\alpha - 5\zeta)\mu(Q_k(t, (1-\zeta)r)) - 2\text{Lip}(f)\mu(Q_k(t, r) \setminus \gamma^{-1}(E)) \\
& \geq 3\alpha\mu(Q_k(t, r))/4 - 2\text{Lip}(f)\alpha\mu(Q_k(t, r))/8\text{Lip}(f) \\
& \geq \alpha\mu(Q_k(t, r))/2. \quad \square
\end{aligned}$$

We now extend the notation introduced in the beginning of Section 2 for  $\gamma \in \Gamma(X)$  to the more general setting of  $L_1(T, X)$ . For  $k \in \mathbb{N}$  and  $s \in T$ ,  $g \in L_1(T, X)$  we put

$$g^{k,s}(t) = g(t_1, \dots, t_k, s_{k+1}, \dots).$$

With this notation, the Fubini's formula says that for every  $k$ ,

$$\int_T g(t) d\mu(t) = \int_T \int_T g^{k,s}(t) d\mu(s) d\mu(t) = \int_T \int_T g^{k,s}(t) d\mu(t) d\mu(s).$$

As in the case of continuous functions, the functions  $g^{k,s}$  approximate  $g$  for large enough  $k$ . The precise formulation of this statement is given in

LEMMA 3.6. *Suppose that  $g \in L_1(T, X)$ . Then for any  $\eta > 0$  there is an  $l \in \mathbb{N}$  such that*

$$\mu\{s \in T : \|g^{k,s} - g\|_{L_1} > \eta\} < \eta$$

for  $k \geq l$ .

*Proof.* Let  $\tilde{g} : T \rightarrow X$  be a continuous function depending on the first  $l$  variables only such that  $\|g - \tilde{g}\|_{L_1} < \eta^2$  and let  $k \geq l$ . By Fubini's theorem,

$$\int_T \|g^{k,s} - \tilde{g}\|_{L_1} d\mu(s) = \|g - \tilde{g}\|_{L_1} < \eta^2$$

so by Chebyshev's inequality

$$\mu\{s \in T : \|g^{k,s} - \tilde{g}\|_{L_1} > \eta\} < \eta$$

as desired.  $\square$

The next lemma is a version of Lebesgue's differentiability theorem for functions defined on the infinite torus  $T$ .

LEMMA 3.7. *Suppose that  $g \in L_1(T, X)$ . Then for every  $\kappa > 0$  there is an  $l \in \mathbb{N}$  such that for all  $k \geq l$  there are  $\delta > 0$  and  $A \subset T$  with  $\mu(A) < \kappa$  such that*

$$\int_{Q_k(t,r)} \|g(s) - g(t)\| d\mu(s) < \kappa\mu(Q_k(t, r))$$

for every  $t \in T \setminus A$  and  $0 < r < \delta$ .

*Proof.* We find it convenient to use the notation

$$\oint_{Q_k(t,r)} h(s) d\mu(s) \quad \text{for} \quad \mu(Q_k(t,r))^{-1} \int_{Q_k(t,r)} h(s) d\mu(s).$$

Let  $\tilde{g} : T \rightarrow X$  be a continuous function depending on the first  $l$  coordinates so that  $\|g - \tilde{g}\|_{L_1} < \kappa^2/9$ . If we put  $N = \{t : \|g(t) - \tilde{g}(t)\| \geq \kappa/3\}$ , we get by Chebyshev's inequality that  $\mu(N) < \kappa/3$ .

Fix an arbitrary  $k \geq l$  and put  $h(s) = \int \|g^{k,\sigma}(s) - \tilde{g}(s)\| d\mu(\sigma)$  and  $S = \{s : h(s) \geq \kappa/3\}$ . Then as in the proof of Lemma 3.6 we get from Fubini's theorem that  $\mu(S) < \kappa/3$ .

Let

$$A_n = \{t \notin S : \oint_{Q_k(t,r)} h(s) d\mu(s) \geq \kappa/3 \text{ for some } 0 < r < 1/n\}.$$

By Lebesgue's differentiability theorem,

$$\lim_{r \rightarrow 0} \oint_{Q_k(t,r)} h(s) d\mu(s) = h(t)$$

for  $\mu$ -almost every  $t$ . Hence  $A_n$  is a decreasing sequence of measurable sets whose intersection has measure zero. Let  $n_0$  be such that  $\mu(A_{n_0}) < \kappa/3$  and put  $A = N \cup S \cup A_{n_0}$ . Choose  $0 < \delta < 1/n_0$  such that  $\|\tilde{g}(s) - \tilde{g}(t)\| < \kappa/3$  whenever  $s \in Q_k(t, \delta)$ . Then  $\mu(A) < \kappa$  and for every  $t \in T$  and  $0 < r < \delta$  we have, by Fubini's theorem and the fact that  $\tilde{g}$  depends on the first  $k$  coordinates, that

$$\begin{aligned} \int_{Q_k(t,r)} \|g(s) - \tilde{g}(s)\| d\mu(s) &= \int_{Q_k(t,r)} \int_T \|g^{k,\sigma}(s) - \tilde{g}(s)\| d\mu(\sigma) d\mu(s) \\ &= \int_{Q_k(t,r)} h(s) d\mu(s). \end{aligned}$$

Hence, for every  $t \in T \setminus A$  and  $0 < r < \delta$ ,

$$\begin{aligned} \oint_{Q_k(t,r)} \|g(s) - g(t)\| d\mu(s) &\leq \oint_{Q_k(t,r)} \|g(s) - \tilde{g}(s)\| d\mu(s) + \|g(t) - \tilde{g}(t)\| \\ &\quad + \oint_{Q_k(t,r)} \|\tilde{g}(s) - \tilde{g}(t)\| d\mu(s) \\ &< \kappa/3 + \kappa/3 + \kappa/3 = \kappa. \end{aligned} \quad \square$$

Our next lemma is in the spirit of descriptive set theory (see e.g. [6] for background). It is this lemma which makes it clear why the separability assumption on  $\mathcal{L}$  in the statement of the main theorem (Theorem 3.10 below) is needed.

Before stating the lemma, we list some assumptions and definitions which enter into its statement. We assume that  $X$  and  $Y$  are separable Banach spaces. The function  $f$  is a Lipschitz function from an open set  $G \subset X$  into  $Y$ . We let  $\mathcal{L}$  be a norm separable subspace of the space  $\text{Lin}(X, Y)$  of bounded

linear operators from  $X$  to  $Y$ . We let  $E$  be the set of those points  $x$  in  $G$  at which  $f$  is regular and Gâteaux differentiable and  $D_f(x) \in \mathcal{L}$ . We denote by  $\varphi$  the characteristic function of  $E$  (as a subset of  $X$ ) and let  $\psi(x) = D_f(x)$  for  $x \in E$  and  $\psi(x) = 0$  if  $x \notin E$ . We also put  $\Phi(\gamma) = \varphi \circ \gamma$  and  $\Psi(\gamma) = \psi \circ \gamma$ .

**LEMMA 3.8.** *The set  $E$  is a Borel set and the mappings  $\varphi : X \rightarrow \mathbb{R}$ ,  $\psi : X \rightarrow \mathcal{L}$ ,  $\Phi : \Gamma(X) \rightarrow L_1(T)$  and  $\Psi : \Gamma(X) \rightarrow L_1(T, \mathcal{L})$  are all Borel measurable. In particular there is a residual subset  $H$  of  $\Gamma(X)$  such that the restrictions of  $\Phi$  and  $\Psi$  to  $H$  are continuous.*

*Proof.* For  $L \in \mathcal{L}$ ,  $u, v \in X$  and  $\sigma, \tau > 0$  denote by  $M(L, u, v, \sigma, \tau)$  the set of  $x \in G$  such that  $\text{dist}(x, X \setminus G) \geq \tau(\|u\| + \|v\|)$  and  $\|f(x + su + sv) - f(x + sv) - sL(u)\| \leq |s|\sigma\|u\|$  whenever  $|s| < \tau$ . Clearly each  $M(L, u, v, \sigma, \tau)$  is a closed subset of  $X$ .

Let  $\mathcal{S}$  be a countable dense subset of  $\mathcal{L}$ ,  $D$  a dense countable subset of  $X$ , and  $R$  be the set of positive rational numbers. Then

$$E = \bigcap_{\sigma \in R} \bigcup_{L \in \mathcal{S}} \bigcap_{u \in D} \bigcup_{\tau \in R} \bigcap_{v \in D, \|v\| \leq 1} M(L, u, v, \sigma, \tau)$$

and hence  $E$  is Borel and  $\varphi$  is Borel measurable.

For every  $L \in \mathcal{L}$  and  $\varrho > 0$  we have

$$\{x \in E : \|\psi(x) - L\| \leq \varrho\} = E \cap \bigcap_{u \in D} \bigcap_{R \ni \sigma > \varrho} \bigcup_{\tau \in R} M(L, u, 0, \sigma, \tau).$$

Since  $E$  is Borel,  $\mathcal{L}$  is separable and  $\psi(x) = 0$  outside  $E$ , it follows that also  $\psi$  is Borel measurable.

Since  $\psi$  is bounded and Borel measurable, the Borel measurability of  $\Psi$  will be established once we show that for every Borel measurable bounded  $h : X \rightarrow \mathcal{L}$  the mapping  $\Psi_h : \Gamma(X) \rightarrow L_1(T, \mathcal{L})$  defined by  $\Psi_h(\gamma) = h \circ \gamma$  is Borel measurable. If  $h$  is continuous, then so is  $\Psi_h$ . If  $\{h_n\}_{n=1}^\infty$  are uniformly bounded in norm and  $h_n \rightarrow h$  pointwise and if all  $\Psi_{h_n}$  are Borel measurable, then  $\Psi_{h_n}$  converge pointwise to  $\Psi_h$ , so  $\Psi_h$  is Borel measurable. The same argument shows the Borel measurability of  $\Phi$ .

The last statement in the lemma follows from the general fact that a Borel measurable mapping between complete separable metrizable spaces has a continuous restriction to a suitable residual subset.  $\square$

*Remark.* Without assuming the separability of  $\mathcal{L}$  not only does the proof not work but the statement is actually false. The Borel image of a complete separable metric space is again separable. Hence if  $\Psi$  is Borel, the image of  $\Gamma(X)$  under  $\Psi$  must be separable.

The final lemma before the proof of the main theorem combines much of the previous lemmas. According to Lemma 3.5, if  $f$  is regular but not Fréchet

differentiable at a point of a surface then a suitable small local deformation of the surface causes a nonsmall perturbation of the function  $t \rightarrow D_f(\gamma(t))$ . Here we perform a finite number of such local perturbations (on suitable disjoint neighborhoods chosen via the Vitali theorem) and get the same effect globally. The notation is as in the preceding lemmas.

LEMMA 3.9. *Suppose that  $\varepsilon, \eta > 0$  and that  $\tilde{\gamma} \in \Gamma_l(X)$  is such that the set*

$$S = \{t \in \tilde{\gamma}^{-1}(E) : \limsup_{u \rightarrow 0} \|f(\tilde{\gamma}(t) + u) - f(\tilde{\gamma}(t)) - D_f(\tilde{\gamma}(t))u\|/\|u\| > \varepsilon\}$$

*has  $\mu$  measure greater than  $\varepsilon$ . Then there are  $k \geq l$ ,  $\delta > 0$  and  $\hat{\gamma} \in \Gamma_{k+1}(X)$  such that  $\|\hat{\gamma} - \tilde{\gamma}\|_{\leq k} < \eta$  and*

$$\int_T (|\varphi(\gamma(t)) - \varphi(\tilde{\gamma}(t))| + \|\psi(\gamma(t)) - \psi(\tilde{\gamma}(t))\|) d\mu(t) > \varepsilon^2/8(1 + 4\text{Lip}(f))$$

*whenever  $\gamma \in \Gamma(X)$  and  $\|\gamma - \hat{\gamma}\|_{\leq k+1} < \delta$ .*

*Proof.* By Lemma 3.7 we can find  $k \geq l$ ,  $\delta_0 > 0$  and a set  $A \subset T$  with  $\mu(A) < \varepsilon/4$  such that for every  $t \in T \setminus A$  and  $0 < r < \delta_0$

$$(7) \quad \int_{Q_k(t,r)} |\varphi(\tilde{\gamma}(s)) - \varphi(\tilde{\gamma}(t))| d\mu(s) < \varepsilon\mu(Q_k(t,r))/16\text{Lip}(f),$$

and

$$(8) \quad \int_{Q_k(t,r)} \|\psi(\tilde{\gamma}(s)) - \psi(\tilde{\gamma}(t))\| d\mu(s) < \varepsilon\mu(Q_k(t,r))/4.$$

For every  $t \in S \setminus A$  and  $n \in \mathbb{N}$  we use Lemma 3.5 with

$$\eta_n = \min(\eta, \delta_0)/(n+1)$$

to find  $0 < r_{t,n} < \eta_n$ ,  $\delta_{t,n} > 0$  and  $\hat{\gamma}_{t,n} \in \Gamma_{k+1}(X)$  such that  $\|\hat{\gamma}_{t,n} - \tilde{\gamma}\|_{\leq k} < \eta_n$ ,  $\hat{\gamma}_{t,n}(s) = \tilde{\gamma}(s)$  for  $s \in T \setminus Q_k(t, r_{t,n})$  and either

$$(9) \quad \mu(Q_k(t, r_{t,n}) \setminus \gamma^{-1}(E)) \geq \varepsilon\mu(Q_k(t, r_{t,n}))/8\text{Lip}(f)$$

or

$$(10) \quad \int_{Q_k(t, r_{t,n}) \cap \gamma^{-1}(E)} \|D_f(\gamma(s)) - D_f(\tilde{\gamma}(t))\| d\mu(s) \geq \varepsilon\mu(Q_k(t, r_{t,n}))/2,$$

whenever  $\gamma \in \Gamma(X)$  and  $\|\gamma(s) - \hat{\gamma}_{t,n}(s)\| + \|D_{k+1}(\gamma(s) - \hat{\gamma}_{t,n}(s))\| < \delta_{t,n}$  for all  $s \in Q_k(t, r_{t,n})$ .

The reason to introduce here the extra parameter  $n$  is to enable us to use the Vitali covering theorem later on (see e.g. the definition of  $\eta_n$ ).

If (9) holds, it follows using (7) and the fact that  $\varphi(\tilde{\gamma}(t)) = 1$  and that  $\varphi$  vanishes outside  $\gamma^{-1}(E)$  that



$$\begin{aligned}
& \int_{Q_k(t, r_{t,n})} |\varphi(\gamma(s)) - \varphi(\tilde{\gamma}(s))| d\mu(s) \\
& \geq \int_{Q_k(t, r_{t,n})} |\varphi(\gamma(s)) - \varphi(\tilde{\gamma}(t))| d\mu(s) - \int_{Q_k(t, r_{t,n})} |\varphi(\tilde{\gamma}(s)) - \varphi(\tilde{\gamma}(t))| d\mu(s) \\
& > \varepsilon \mu(Q_k(t, r_{t,n}))/8\text{Lip}(f) - \varepsilon \mu(Q_k(t, r_{t,n}))/16\text{Lip}(f) \\
& = \varepsilon \mu(Q_k(t, r_{t,n}))/16\text{Lip}(f).
\end{aligned}$$

If (10) holds, we get using (8) that

$$\int_{Q_k(t, r_{t,n})} \|\psi(\gamma(s)) - \psi(\tilde{\gamma}(s))\| d\mu(s) > \varepsilon \mu(Q_k(t, r_{t,n}))/4.$$

Hence, in any case,

$$\begin{aligned}
(11) \quad & \int_{Q_k(t, r_{t,n})} (|\varphi(\gamma(s)) - \varphi(\tilde{\gamma}(s))| + \|\psi(\gamma(s)) - \psi(\tilde{\gamma}(s))\|) d\mu(s) \\
& > \varepsilon \mu(Q_k(t, r_{t,n}))/4(1 + 4\text{Lip}(f))
\end{aligned}$$

for all  $t \in S \setminus A$ ,  $n \in \mathbb{N}$  and  $\gamma \in \Gamma(X)$  such that

$$\|\gamma(s) - \hat{\gamma}_{t,n}(s)\| + \|D_{k+1}(\gamma(s) - \hat{\gamma}_{t,n}(s))\| < \delta_{t,n}$$

for all  $s \in Q_k(t, r_{t,n})$ .

By the Vitali covering theorem, there are  $t_1, \dots, t_j$  and  $n_1, \dots, n_j$  such that  $\{Q_k(t_i, r_{t_i, n_i})\}_{1 \leq i \leq j}$  are disjoint and cover the set  $S \setminus A$  up to a set of measure less than  $\varepsilon/4$ . Define now  $\hat{\gamma}(s) = \hat{\gamma}_{t_i, n_i}(s)$  if  $s \in Q_k(t_i, r_{t_i, n_i})$ ,  $1 \leq i \leq j$ , and  $\hat{\gamma}(s) = \tilde{\gamma}(s)$  otherwise. Clearly  $\|\hat{\gamma} - \tilde{\gamma}\|_{\leq k} < \eta$ . Letting  $\delta = \min_{1 \leq i \leq j} \delta_{t_i, n_i}$  then for every  $\gamma \in \Gamma(X)$  with  $\|\gamma - \hat{\gamma}\|_{\leq k+1} < \delta$  we get by adding (11) over all the  $j$  cubes  $Q_k(t_i, r_{t_i, n_i})$  that

$$\begin{aligned}
& \int_T (|\varphi(\gamma(s)) - \varphi(\tilde{\gamma}(s))| + \|\psi(\gamma(s)) - \psi(\tilde{\gamma}(s))\|) d\mu(s) \\
& > \varepsilon(\mu(S) - \varepsilon/2)/4(1 + 4\text{Lip}(f)) \geq \varepsilon^2/8(1 + 4\text{Lip}(f)). \quad \square
\end{aligned}$$

We are now ready to prove the main theorem.

**THEOREM 3.10.** *Suppose that  $G$  is an open subset of a separable Banach space  $X$ ,  $\mathcal{L}$  a norm separable subspace of  $\text{Lin}(X, Y)$ , and  $f : G \rightarrow Y$  is a Lipschitz function. Then  $f$  is Fréchet differentiable at  $\Gamma$ -almost every point  $x \in X$  at which it is regular, Gâteaux differentiable and  $D_f(x) \in \mathcal{L}$ .*

*Proof.* We may assume that  $Y$  is separable and continue to use the notation of the previous lemmas. By Lemma 3.8 there is a residual subset  $H$  of  $\Gamma(X)$  such that the restrictions of  $\Phi$  and  $\Psi$  to  $H$  are continuous.

Fix  $\varepsilon > 0$  and put

$$N = \{x \in E : \limsup_{u \rightarrow 0} \|f(x+u) - f(x) - D_f(x)u\|/\|u\| > \varepsilon\}.$$

To prove the theorem it suffices to show that the set

$$M = \{\gamma \in H : \mu\{t : \gamma(t) \in N\} > 2\varepsilon\}$$

is nowhere dense in  $\Gamma(X)$ . Assume that this is not the case, then we can find a nonempty open set  $U$  in the closure of  $M$ . Let  $\gamma_0 \in M \cap U$  and making  $U$  smaller if necessary we can achieve that

$$(12) \quad \|\Phi(\gamma) - \Phi(\gamma_0)\|_{L_1} + \|\Psi(\gamma) - \Psi(\gamma_0)\|_{L_1} < \varepsilon^2/16(1 + 4\text{Lip}(f))$$

for every  $\gamma \in U \cap H$ .

Let  $l_0 \in \mathbb{N}$  and  $\eta_0 > 0$  be such that every  $\gamma$  with  $\|\gamma - \gamma_0\|_{\leq l_0} < 3\eta_0$  belongs to  $U$ . By Lemma 3.6 we can find  $l \geq l_0$  so that

- $\|\gamma_0^{l,s} - \gamma_0\|_{\leq l_0} < \eta_0$  for every  $s \in T$ ,
- $\mu\{s \in T : \|\Phi(\gamma_0^{l,s}) - \Phi(\gamma_0)\|_{L_1} \geq \varepsilon^2/32(1 + \text{Lip}(f))\} < \varepsilon/2$ , and
- $\mu\{s \in T : \|\Psi(\gamma_0^{l,s}) - \Psi(\gamma_0)\|_{L_1} \geq \varepsilon^2/32(1 + \text{Lip}(f))\} < \varepsilon/2$ .

By Fubini's theorem, there is  $s \in T$  such that  $\tilde{\gamma} = \gamma_0^{l,s}$  has the properties that  $\mu\{t : \tilde{\gamma}(t) \in N\} > \varepsilon$  and

$$(13) \quad \|\Phi(\tilde{\gamma}) - \Phi(\gamma_0)\|_{L_1} + \|\Psi(\tilde{\gamma}) - \Psi(\gamma_0)\|_{L_1} < \varepsilon^2/16(1 + \text{Lip}(f)).$$

By Lemma 3.9, we can find  $k \geq l$ ,  $\delta > 0$  and  $\hat{\gamma} \in \Gamma_{k+1}(X)$  such that  $\|\hat{\gamma} - \tilde{\gamma}\|_{\leq k} < \eta_0$  and

$$(14) \quad \int (|\varphi(\gamma(t)) - \varphi(\tilde{\gamma}(t))| + \|\psi(\gamma(t)) - \psi(\tilde{\gamma}(t))\|) d\mu(t) > \varepsilon^2/8(1 + 4\text{Lip}(f))$$

whenever  $\gamma \in \Gamma(X)$  and  $\|\gamma - \hat{\gamma}\|_{\leq k+1} < \delta$ . There are  $\gamma \in U \cap H$  with  $\|\gamma - \hat{\gamma}\|_{\leq k+1} < \delta$ , and for any such  $\gamma$  both (13) and (14) hold; hence

$$\|\Phi(\gamma) - \Phi(\gamma_0)\|_{L_1} + \|\Psi(\gamma) - \Psi(\gamma_0)\|_{L_1} > \varepsilon^2/16(1 + 4\text{Lip}(f)).$$

This contradicts (12) and thus the assumption that the closure of  $M$  contains a nonempty open subset led to a contradiction.  $\square$

**COROLLARY 3.11.** *Assume that  $X^*$  is separable. Then any convex continuous function on an open subset of  $X$  is  $\Gamma$ -almost everywhere Fréchet differentiable.*

*Proof.* This is an immediate consequence of Theorem 2.5, Proposition 3.2 and Theorem 3.10.  $\square$

**COROLLARY 3.12.** *Assume that  $X^*$  is separable. Then any Lipschitz function  $f$  from an open subset  $G$  of  $X$  into  $\mathbb{R}$  is  $\Gamma$ -almost everywhere Fréchet differentiable if and only if every  $\sigma$  porous set in  $X$  is  $\Gamma$ -null.*

*Proof.* The “if” part is an immediate consequence of Theorem 2.5, Proposition 3.3 and Theorem 3.10.

The “only if” part is trivial: As already remarked in Section 1, if  $A$  is porous, the Lipschitz function  $f(x) = \text{dist}(x, A)$  is nowhere Fréchet differentiable on  $A$ .  $\square$

#### 4. Spaces in which $\sigma$ -porous sets are $\Gamma$ -null

In this section we present the second main result of this paper in which we identify a class of Banach spaces in which  $\sigma$ -porous sets are  $\Gamma$ -null. The basic observation behind these results is that, if a surface passes through a point of a porous set  $E$ , then its suitable small deformation passes through the center of a relatively large ball that completely avoids  $E$ . This process may be iterated in order to construct surfaces avoiding more and more of  $E$ . Unfortunately, the final deformation cannot be guaranteed to remain small. As we shall see in Corollary 5.3, in  $\ell_2$  this problem is unsurmountable. However, in  $c_0$  any combination of small perturbations is still small provided that different perturbations use disjoint sets of coordinates, and the idea can be used to show that sets porous in the direction of all subspaces  $\{x \in c_0 : x_1 = \dots = x_k = 0\}$  are  $\Gamma$ -null. A simple decomposition of porous sets (Lemma 4.3) will then finish the proof that every porous subset of  $c_0$  is  $\Gamma$ -null. Moreover, a close inspection of the first argument reveals that the structure of  $c_0$  is needed only asymptotically (in the sense of the following definition), which will enable us to extend the above arguments to several other spaces.

*Definition 4.1.* Let  $X$  be a Banach space and  $\{X_k\}_{k=1}^\infty$  a decreasing sequence of subspaces of  $X$ . The sequence  $X_k$  of subspaces is said to be asymptotically  $c_0$  if there is  $C < \infty$  so that for every  $n \in \mathbb{N}$ ,

$$(\exists k_1 \in \mathbb{N})(\forall u_1 \in X_{k_1})(\exists k_2 \in \mathbb{N})(\forall u_2 \in X_{k_2}) \dots (\exists k_n \in \mathbb{N})(\forall u_n \in X_{k_n}) \\ \|u_1 + \dots + u_n\| \leq C \max(\|u_1\|, \dots, \|u_n\|).$$

A formally stronger requirement than that of Definition 4.1 which is however easily seen to be equivalent to it is that there is  $C < \infty$  such that for every  $n \in \mathbb{N}$ ,

$$(15) \quad (\exists k_1 \in \mathbb{N})(\forall U_1 \subset\subset X_{k_1}) \dots (\exists k_n \in \mathbb{N})(\forall U_n \subset\subset X_{k_n}) \\ (\forall u_i \in U_i) \|u_1 + \dots + u_n\| \leq C \max(\|u_1\|, \dots, \|u_n\|),$$

where the symbol  $\subset\subset$  means “a finite-dimensional subspace of”.

The main part of the proofs of the results of this section is contained in the following lemma.

LEMMA 4.2. *Suppose that a sequence  $\{X_k\}_{k=1}^\infty$  of subspaces of  $X$  is asymptotically  $c_0$ . Then for every  $c > 0$  every set  $E \subset X$  which is  $c$ -porous in the direction of all the subspaces  $X_k$  is  $\Gamma$ -null.*

The notion of a set being  $c$ -porous in the direction of a subspace is defined in Section 1.

*Proof.* It suffices to find a contradiction from the assumption that there is a nonempty open subset  $H$  of  $\Gamma(X)$  having the property that, for some  $\varepsilon > 0$ , every nonempty open subset  $G$  of  $H$  contains a  $\gamma \in \Gamma(X)$  such that

$$(16) \quad \mu\{t : \gamma(t) \in E\} > \varepsilon.$$

Let  $\tilde{\gamma}_1 \in H$  be such that  $\mu\{t : \tilde{\gamma}_1(t) \in E\} > \varepsilon$ . Find  $\tilde{m} \in \mathbb{N}$  and  $\delta_1 > 0$  such that every  $\gamma \in \Gamma(X)$  satisfying  $\|\gamma - \tilde{\gamma}_1\|_{\leq \tilde{m}} < 2\delta_1$  belongs to  $H$ . We fix  $m \geq \tilde{m}$  such that  $\|\tilde{\gamma}_1^{m,t} - \tilde{\gamma}_1\|_{\leq \tilde{m}} < \delta_1$  for all  $t \in T$  and use Fubini's theorem to find  $\bar{t} \in T$  such that the surface  $\gamma_1 = \tilde{\gamma}_1^{m,\bar{t}}$  satisfies  $\mu\{t : \gamma_1(t) \in E\} > \varepsilon$ . We choose  $M < \infty$  with the property that for every  $\gamma \in \Gamma_m(X)$  such that  $\|\gamma - \gamma_1\|_{\leq m} \leq \delta_1$ ,

$$(17) \quad \|\gamma(t) - \gamma(s)\| \leq M \max_{1 \leq j \leq m} |t_j - s_j| \text{ for all } t, s \in T.$$

Denote  $\kappa = c/4M$  and  $K = 4\max(\kappa, C/\delta_1)$ , where  $C$  is the constant for which (15) holds and denote

$$Q(s, r) = \{t \in \ell_\infty : |t_j - s_j| \leq r \text{ for } j = 1, \dots, m\}.$$

Choose  $n \in \mathbb{N}$  so that

$$(18) \quad (n-1)\varepsilon(\kappa/K)^m > 2.$$

It is this  $n$  for which we intend to use (15).

For  $i = 1, \dots, n$  we will define inductively indices  $k_i \in \mathbb{N}$ , surfaces  $\gamma_i, \psi_i \in \Gamma_m(X)$ , finite sets  $S_i \subset T$ , finite-dimensional subspaces  $U_i$  of  $X_{k_i}$ , sets  $W_i, Q_i \subset T$ , and numbers  $\delta_i > 0$  so that in particular the following statements hold for  $1 \leq i \leq n$ :

$$(19) \quad \mu(W_i) > \varepsilon$$

$$(20) \quad B(\gamma_i(s) + \psi_i(s), c\|\psi_i(s)\|) \cap E = \emptyset \text{ if } s \in S_i$$

and the following statements hold for  $1 \leq i \leq n-1$ :

$$(21) \quad \delta_{i+1} = \frac{1}{4} \min_{1 \leq l \leq i} (\delta_l, c \min_{s \in S_l} \|\psi_l(s)\|),$$

$$(22) \quad \|\gamma_{i+1} - \gamma_i\|_0 < \delta_i,$$

$$(23) \quad \|\gamma_{i+1} - (\gamma_i + \psi_i)\|_{\leq m} < \delta_{i+1},$$

$$(24) \quad \|\gamma_{i+1} - \gamma_1\|_{\leq m} < \delta_1.$$

Assume that for some  $1 \leq i \leq n$  we have already defined  $\gamma_i, \delta_i$  and all  $\gamma_j, \delta_j, k_j, \psi_j, W_j, Q_j, S_j, U_j$  for  $1 \leq j < i$ . Since  $\gamma_1$  and  $\delta_1$  have been already defined, this is certainly true for  $i = 1$ . We show how to choose  $k_i, W_i, Q_i, S_i, U_i, \psi_i$  and, provided that  $i < n$ ,  $\delta_{i+1}$  and  $\gamma_{i+1}$ .

We find  $k_i \in \mathbb{N}$  as in (15). Let  $W_i = \{t \in T : \gamma_i(t) \in E\}$  and

$$Q_i = \bigcup_{l=1}^{i-1} \bigcup_{s \in S_l} Q(s, \kappa \|\psi_l(s)\|).$$

The  $\gamma_i$  has been chosen from among the surfaces for which (16) holds, hence  $\mu(W_i) > \varepsilon$  as required by (19). We show next that  $W_i \cap Q_i = \emptyset$ . Since  $Q_1 = \emptyset$ , there is nothing to prove for  $i = 1$ . Let  $i > 1$  and  $t \in Q(s, \kappa \|\psi_l(s)\|)$  for some  $1 \leq l < i$ . We deduce from (24) that (17) is applicable to  $\gamma_i$  and infer, using also (22) and (23), that

$$\begin{aligned} & \|\gamma_i(t) - (\gamma_l(s) + \psi_l(s))\| \\ & \leq \|\gamma_i(t) - \gamma_i(s)\| + \|\gamma_i(s) - \gamma_{l+1}(s)\| + \|\gamma_{l+1}(s) - (\gamma_l(s) + \psi_l(s))\| \\ & \leq M\kappa\|\psi_l(s)\| + \sum_{j=l+1}^{i-1} \delta_j + \delta_{l+1} \\ & < c\|\psi_l(s)\|. \end{aligned}$$

Hence (20) implies that  $\gamma_i(t) \notin E$  and thus  $t \notin W_i$ .

For every  $t \in W_i$  choose  $u_{j,i}(t) \in X_{k_i}$  such that  $0 < \|u_{j,i}(t)\| < \delta_i/4^j$  and  $B(\gamma_i(t) + u_{j,i}(t), c\|u_{j,i}(t)\|) \cap E = \emptyset$ . This can be done since, by assumption,  $E$  is  $c$ -porous in the direction of  $X_{k_i}$ . Using Vitali's covering theorem (in  $[0, 1]^m$ ), we find a finite set  $S_i \subset W_i$  and for each  $s \in S_i$  vectors  $u_i(s) = u_{j_i,i}(s)$  such that the cubes  $Q(s, K\|u_i(s)\|)$  are disjoint, contained in  $T \setminus Q_i$  and so that

$$(25) \quad \mu\left(\bigcup_{s \in S_i} Q(s, K\|u_i(s)\|)\right) \geq \mu(W_i)/2.$$

We define next  $U_i$  to be the span of  $u_i(s)$ ,  $s \in S_i$ . For each  $s \in S_i$  we choose  $\omega_s \in \Gamma_m(\mathbb{R})$  depending on the first  $m$  coordinates (so essentially  $\omega_s : [0, 1]^m \rightarrow \mathbb{R}$ ) such that  $0 \leq \omega_s \leq 1$  and

$$\begin{aligned} \omega_s(t) &= 0 \text{ for } t \notin Q(s, K\|u_i(s)\|), \\ \omega_s(s) &= 1, \text{ and} \\ \|D_j \omega_s\|_0 &\leq 2/(K\|u_i(s)\|) \text{ for } j = 1, \dots, m. \end{aligned}$$

Let  $\psi_i(t)$  be defined by  $\psi_i(t) = \sum_{s \in S_i} \omega_s(t) u_i(s)$ . For each  $t$ ,  $\psi_i(t) \in U_i$  and the same is true for the partial derivatives of  $\psi_i$ . Since  $\psi_i(s) = u_i(s)$  for  $s \in S_i$ , (20) follows from the choice of the  $u_i(s)$ . The definition of  $u_i(s)$  gives also that  $\|\psi_i\|_0 < \delta_i/4$  and consequently, from (23) and  $\delta_{l+1} \leq \delta_l/4$ ,

$$\begin{aligned}
\|\gamma_i + \psi_i - \gamma_1\|_0 &\leq \left\| \sum_{l=1}^i \psi_l \right\|_0 + \sum_{l=1}^{i-1} \|\gamma_{l+1} - (\gamma_l + \psi_l)\|_0 \\
&\leq 2 \sum_{l=1}^i \delta_l/4 < \delta_1.
\end{aligned}$$

Since  $D_j \psi_l(t) \in U_l$  for all  $1 \leq j \leq m$  and  $t \in T$ , we get from (15) that

$$\|D_j(\sum_{l=1}^i \psi_l)(t)\| \leq C \max_{1 \leq l \leq i} \|D_j \psi_l(t)\| \leq 2C/K.$$

Hence, by (23),

$$\begin{aligned}
(26) \quad \|\gamma_i + \psi_i - \gamma_1\|_j &\leq \left\| \sum_{l=1}^i \psi_l \right\|_j + \sum_{l=1}^{i-1} \|\gamma_{l+1} - (\gamma_l + \psi_l)\|_j \\
&\leq 2C/K + \sum_{l=1}^{i-1} \delta_{l+1} < \delta_1,
\end{aligned}$$

and thus  $\gamma_i + \psi_i \in H$ .

If  $i = n$ , this finishes the construction. If  $i < n$ , we still have to define  $\delta_{i+1}$  and  $\gamma_{i+1}$  and show that (21), (22), (23) and (24) remain valid. Since  $\psi_i(s) \neq 0$ , (21) may be used as a definition of  $\delta_{i+1}$ . By our assumption on  $H$ , there is a  $\tilde{\gamma}_{i+1} \in H$  such that

$$\|\tilde{\gamma}_{i+1} - (\gamma_i + \psi_i)\|_{\leq m} < \delta_{i+1} \text{ and } \mu(\{t : \tilde{\gamma}_{i+1}(t) \in E\}) > \varepsilon.$$

By Fubini's theorem there is a  $\bar{t} \in T$  such that the surface

$$\gamma_{i+1}(t) = \tilde{\gamma}_{i+1}^{m, \bar{t}}$$

satisfies  $\mu\{t : \gamma_{i+1}(t) \in E\} > \varepsilon$ . Since  $\gamma_i$  and  $\psi_i$  belong to  $\Gamma_m(X)$ , it follows that (22) holds. Since  $\|\psi_i\|_0 < \delta_i/4$ , also (23) holds. To establish (24), we use (26) to get

$$\begin{aligned}
\|\gamma_{i+1} - \gamma_1\|_{\leq m} &\leq \|\gamma_{i+1} - (\gamma_i + \psi_i)\|_{\leq m} + \|\gamma_i + \psi_i - \gamma_1\|_{\leq m} \\
&\leq \delta_{i+1} + 2C/K + \sum_{l=1}^{i-1} \delta_{l+1} < \delta_1.
\end{aligned}$$

This finishes the  $i$ -th step of the construction. We now loop back to the choice of  $k_{i+1}$  according to (15) and continue the inductive process.

Finally, by (19) and (25) we get

$$\begin{aligned}
\mu(Q_{i+1}) &= \mu(Q_i) + \sum_{s \in S_i} \mu(Q(s, \kappa \|u_i(s)\|)) \\
&= \mu(Q_i) + (\kappa/K)^m \sum_{s \in S_i} \mu(Q(s, K \|u_i(s)\|)) \\
&\geq \mu(Q_i) + (\kappa/K)^m \varepsilon/2.
\end{aligned}$$

Hence  $\mu(Q_n) \geq (n-1)(\kappa/K)^m \varepsilon/2 > 1$  by our choice of  $n$  in (18) and we get the desired contradiction.  $\square$

In order to apply Lemma 4.2 we need the following lemma which is a variant of Lemma 4.6 from [15].

**LEMMA 4.3.** *Let  $U, V$  be subspaces of a Banach space  $X$  such that for some  $\eta < \infty$  every  $x \in U + V$  may be written as  $x = u + v$  where  $u \in U$ ,  $v \in V$  and  $\max(\|u\|, \|v\|) \leq \eta\|x\|$ . Then, if a set  $E \subset X$  is  $c$ -porous in the direction  $U + V$ , then we can write  $E = A \cup B$ , where  $A$  is  $\sigma$ -porous in direction  $U$  and  $B$  is  $c/2\eta$ -porous in direction  $V$ .*

*Proof.* Denote by  $B_m$  the set of those  $x \in E$  for which there is a  $v \in V$  with  $\|v\| < 1/m$  and  $B(x + v, c\|v\|/2\eta) \cap E = \emptyset$ . Clearly,  $B = \bigcap_{m=1}^{\infty} B_m$  is  $c/2\eta$ -porous in direction  $V$ . Thus it is sufficient to prove that each set  $E \setminus B_m$  is porous in direction  $U$ .

Let  $x \in E \setminus B_m$  and  $0 < \varepsilon < 1/m$ . Since  $E$  is  $c$ -porous in direction  $U + V$ , we can find  $z \in U + V$  such that  $0 < \|z\| < \varepsilon/\eta$  and  $B(x + z, c\|z\|) \cap E = \emptyset$ . Write  $z = u + v$  with  $u \in U$ ,  $v \in V$  and  $\max(\|u\|, \|v\|) \leq \eta\|z\|$ .

It suffices to show that  $B(x + u, c\|u\|/2\eta) \cap (E \setminus B_m) = \emptyset$ . For this assume that  $y \in B(x + u, c\|u\|/2\eta) \cap (E \setminus B_m)$ , then  $B(y + v, c\|v\|/2\eta) \subset B(x + u + v, c\|u + v\|) = B(x + z, c\|z\|) \subset X \setminus E$ . Since  $\|v\| \leq \eta\|z\| < \frac{1}{m}$  this shows that  $y \in B_m$  which contradicts our assumption.  $\square$

We can now give concrete examples of spaces having the property that every  $\sigma$ -porous subset in them is  $\Gamma$ -null.

**PROPOSITION 4.4.** *Assume that  $X$  has a Schauder basis  $\{x_i\}_{i=1}^{\infty}$  such that for a suitable sequence  $\{n_k\}_{k=1}^{\infty}$  the sequence  $X_k = \overline{\text{span}}\{x_i\}_{i>n_k}$  is asymptotically  $c_0$ . Then any  $\sigma$ -porous set  $E$  in  $X$  is  $\Gamma$ -null.*

*Proof.* Let  $U_k = \text{span}\{x_i\}_{i \leq n_k}$ . Then clearly  $X = U_k \oplus X_k$  for every  $k$ . If  $E$  is porous we can write (by Lemma 4.3) for every  $k$ ,  $E = A_k \cup B_k$  where  $A_k$  is  $\sigma$ -porous in the direction  $U_k$  and  $B_k$  is  $c_1$ -porous in the direction  $X_k$  for some fixed  $0 < c_1 < 1$ . By Lemma 4.2,  $\bigcap_{k=1}^{\infty} B_k$  is  $\Gamma$ -null. It remains to show that every porous (and thus  $\sigma$ -porous) set  $A$  in direction of a finite-dimensional subspace is  $\Gamma$ -null. A simple compactness argument shows that such a set  $A$  is actually directionally porous. Hence the Lipschitz function from  $X$  to  $\mathbb{R}$  defined by  $f(x) = \text{dist}(x, A)$  is not Gâteaux differentiable at any point of  $A$ . By Theorem 2.5  $A$  is  $\Gamma$ -null.  $\square$

We recall now (see e.g. [5, Lemma 2.13]) the simple fact that if  $W$  is a finite co-dimensional subspace of a Banach space  $X$ , then there is a finite-dimensional subspace  $V$  of  $X$  so that every  $x \in X$  can be written as  $x = v + w$

with  $v \in V$ ,  $w \in W$  and  $\|v\| \leq 2\|x\|$ ,  $\|w\| \leq 3\|x\|$ . By using this fact the argument of Proposition 4.4 can be carried through for any space admitting an asymptotically  $c_0$  sequence of finite co-dimensional subspaces. In particular, any subspace of  $c_0$  also has the property that any porous set in it is  $\Gamma$ -null.

The next lemma allows us to do some iteration arguments concerning the property we are interested in here.

LEMMA 4.5. *Consider the following property of a Banach space  $X$ .*

- ( $\star$ ) *Whenever  $X$  is a complemented subspace of  $Y$  and  $E \subset Y$  is  $\sigma$ -porous in the direction  $X$ , then  $E$  is  $\Gamma$ -null.*

*Any finite or  $c_0$  (infinite) direct sum of spaces having ( $\star$ ) also has ( $\star$ ).*

*Moreover, if there are subspaces  $U_k, V_k$  of  $X$  such that  $U_k$  are complemented in  $X$  and have property ( $\star$ ) while the sequence  $V_k$  is asymptotically  $c_0$  and there is  $\eta < \infty$  such that every  $x \in X$  can be written as  $x = u + v$  where  $u \in U_k$ ,  $v \in V_k$  and  $\max(\|u\|, \|v\|) \leq \eta\|x\|$ , then  $X$  has property ( $\star$ ).*

*Proof.* Assume that  $X_1$  and  $X_2$  have ( $\star$ ),  $X = X_1 \oplus X_2$  is complemented in  $Y$  and  $E \subset Y$  is  $\sigma$ -porous in the direction  $X$ . By Lemma 4.3,  $E = E_1 \cup E_2$ , where  $E_i$  is  $\sigma$ -porous in the direction  $X_i$ . It follows from ( $\star$ ) that each  $E_i$  is  $\Gamma$ -null and thus  $E$  is  $\Gamma$ -null. Hence ( $\star$ ) is stable under finite direct sums.

We prove next the “moreover” statement. This statement and the stability of ( $\star$ ) under finite direct sums imply that ( $\star$ ) is also closed under  $c_0$  direct sums.

Let  $E$  be a  $c$ -porous subset of  $Y$  in the direction  $X$ . By Lemma 4.3,  $E = A_k \cup B_k$ , where  $A_k$  is  $\sigma$ -porous in the direction  $U_k$  and  $B_k$  is  $c/2\eta$ -porous in the direction  $V_k$ . By assumption the  $A_k$  are  $\Gamma$ -null. Since the sequence  $V_k$  is asymptotically  $c_0$ , and the set  $B = \bigcap_{k=1}^{\infty} B_k$  is  $c/2\eta$ -porous in the direction of every  $V_k$ , it follows from Lemma 4.2 that  $B$  is  $\Gamma$ -null and so is  $E = B \cup \bigcup_{k=1}^{\infty} A_k$ .  $\square$

Recall that any compact countable Hausdorff topological space  $K$  is homeomorphic to the space  $K_\alpha$  of ordinals  $\leq \alpha$  in the order topology for some countable ordinal  $\alpha$ . Recall also that  $C(K_\omega)$  is isomorphic to  $c_0$  and that every  $C(K_\alpha)$  is isomorphic to  $(C(K_{\beta_1}) \oplus \dots \oplus C(K_{\beta_k}) \dots)_{c_0}$  for suitable  $\{\beta_k\}_{k=1}^{\infty}$  smaller than  $\alpha$ . Hence we get from Lemma 4.5 by (countable) transfinite induction that each such  $C(K)$  has property ( $\star$ ).

Summing up the preceding observations we get

THEOREM 4.6. *The following spaces have the property that all their  $\sigma$ -porous subsets are  $\Gamma$ -null:  $c_0$ ,  $C(K)$  with  $K$  compact countable, the Tsirelson space, subspaces of  $c_0$ .*



The Tsirelson space (as first defined in [16]) is a space with an unconditional basis which satisfies the assumption in Proposition 4.4. An important feature of this space (in general, and for us here) is that it is reflexive.

From Corollary 3.12 we see that each of the spaces listed in Theorem 4.6 has the property that every real-valued Lipschitz function defined on its open subset is Fréchet differentiable  $\Gamma$ -almost everywhere. Stronger results follow by application of Theorem 3.10. For optimal results in this direction it is desirable to have information when  $\text{Lin}(X, Y)$  is separable for every space  $Y$  with the RNP. Call this property of a Banach space  $(\star\star)$ .

LEMMA 4.7. *The class of spaces  $X$  with the property  $(\star\star)$  is closed under finite direct sums and under infinite direct sums in the sense of  $c_0$ .*

*Proof.* For finite direct sums this is obvious. Assume that  $X = (\bigoplus_{i=1}^{\infty} X_i)_{c_0}$  with each  $X_i$  having  $(\star\star)$ . Denote by  $\pi_k$  the natural projection onto  $\sum_{i=1}^k X_i$  and let  $V_k = (I - \pi_k)X$ ,  $k = 1, 2, \dots$ . Let  $L \in \text{Lin}(X, Y)$  with  $Y$  having the RNP, let  $\varepsilon > 0$  and let  $0 < \eta < 1$  be such that  $\eta/(1 - \eta) < \varepsilon$ .

Since  $Y$  has the RNP, the set  $\{Lz : \|z\| \leq 1\} \subset Y$  has slices with arbitrarily small diameter. Thus there are  $u^* \in U^*$  and a real  $c$  so that the set

$$S = \{Lz : \|z\| \leq 1, u^*(Lz) > c\}$$

is nonempty and of diameter  $< \eta$ . Let  $z \in Z$  with  $\|z\| \leq 1$  be such that  $Lz \in S$  and let  $m$  be such that  $\|z - \pi_m z\| < \eta$ . Whenever  $w \in V_m$  with  $\|w\| \leq 1$  then  $\|z \pm (1 - \eta)w\| \leq 1$ , so at least one of the vectors  $L(z \pm (1 - \eta)w)$  also belongs to  $S$ . Hence  $\|L((1 - \eta)w)\| < \eta$  or  $\|L - L \circ \pi_m\| < \eta/(1 - \eta) < \varepsilon$ . Since  $\text{Lin}(\sum_{i=1}^m X_i, Y)$  is separable for every  $m$ , it follows that  $\text{Lin}(X, Y)$  is separable.  $\square$

Since  $\mathbb{R}$  clearly has  $(\star\star)$ , it follows from Lemma 4.7 that  $c_0$  has  $(\star\star)$  and by countable transfinite induction that  $C(K)$  has  $(\star\star)$  for every compact countable  $K$ .

We show next that every subspace  $Z$  of  $c_0$  has property  $(\star\star)$ . Let  $V_k = \{x \in Z : x_1 = \dots = x_k = 0\}$ . By the observation from [5] used already above there is for every  $k$  a finite-dimensional subspace  $U_k$  of  $Z$  so that every  $z \in Z$  can be written as  $u + v$  with  $u \in U_k$ ,  $v \in V_k$  and  $\|u\|, \|v\| \leq 3\|z\|$ . For every  $L \in \text{Lin}(Z, Y)$  and every  $\varepsilon > 0$  there is a  $k$  such that  $\|Lv\| \leq \varepsilon\|v\|$  for every  $v \in V_k$ . Indeed, if this were false, we would get by a standard gliding bump argument that  $L$  is an isomorphism on a subspace of  $Z$  isomorphic to  $c_0$  which contradicts the assumption that  $Y$  has the RNP.

Since  $U_k$  is finite-dimensional, there is, given  $\varepsilon > 0$ , a countable set  $\mathcal{L}_k \subset \text{Lin}(Z, Y)$  such that for every  $L \in \text{Lin}(Z, Y)$  with  $\|Lv\| \leq \varepsilon\|v\|$  for  $v \in V_k$  there is an  $\hat{L} \in \mathcal{L}_k$  such that  $\|\hat{L}v\| \leq \varepsilon\|v\|$  for  $v \in V_k$  and  $\|\hat{L}u - Lu\| \leq \varepsilon\|u\|$  for  $u \in U_k$ . By decomposing each  $z \in Z$  as above, we deduce that  $\|\hat{L}z - Lz\| \leq$

$6\varepsilon\|z\|$  for every such  $z$ . Hence every  $L \in \text{Lin}(Z, Y)$  has distance  $\leq 6\varepsilon$  from the countable set  $\bigcup_k \mathcal{L}_k$  and this proves the separability of  $\text{Lin}(Z, Y)$ .

From Theorems 3.10 and 4.6 and the preceding arguments we deduce

**THEOREM 4.8.** *The following spaces have the property that every Lipschitz mapping of them into a space with the RNP is Fréchet differentiable  $\Gamma$ -almost everywhere:  $C(K)$  for compact countable  $K$ , subspaces of  $c_0$ .*

*Remark.* Theorem 4.8 does not hold for subspaces of  $C(K)$ ,  $K$  countable. As remarked in [5] the Schreier space which is the completion of the space of eventually zero sequences  $\{a_i\}_{i=1}^\infty$  with respect to the norm

$$\|\{a_i\}\| = \sup\left\{\sum_{k=1}^n |a_{i_k}| : n \in \mathbb{N} \text{ and } n \leq i_1 < i_2 < \dots < i_n\right\}$$

is isomorphic to a subspace of  $C(\omega^\omega)$  and there is a Lipschitz map from this space into  $\ell_2$  which is nowhere Fréchet differentiable.

## 5. The mean value theorem

Let  $X$  be a separable Banach space and  $Y$  a space having the RNP. Let  $D$  be a convex open set in  $X$  and  $D_0$  be the subset of  $D$  consisting of points where the Gâteaux derivative  $D_f(x)$  exists. We know by Theorem 1.1 (resp. Theorem 2.5) that  $D \setminus D_0$  is Gauss null (resp.  $\Gamma$ -null).

Put for  $u \in X$ ,

$$\begin{aligned} R_u &= \left\{ \frac{f(x+tu) - f(x)}{t} : x, x+u \in D \text{ and } t > 0 \right\}, \\ \tilde{R}_u &= \{D_f(x)u : x \in D_0\}. \end{aligned}$$

Then a simple application of the separation theorem, Theorem 1.1 and the property of Gauss null sets that whenever  $x, x+tu \in D$  then there is a point  $x_0$  arbitrarily close to  $x$  such that  $x_0 + su \in D$  for almost all  $0 \leq s \leq t$ , shows that the closed convex hull of  $\tilde{R}_u$  is equal to the closed convex hull of  $R_u$  (see e.g. [5, Lemma 2.12]).

The proofs in [12] and [8] of existence of points of Fréchet differentiability of Lipschitz functions  $f : X \rightarrow \mathbb{R}$  where  $X^*$  is separable actually show the following stronger assertion: Let  $D$  be a convex open set in  $X$  and let  $u, v \in D$  and a constant  $m$  be such that  $f(v) - f(u) > m$ . Then there is a point  $x \in D$  in which  $f$  is Fréchet differentiable and  $D_f(x)(v - u) > m$ .

In view of the observation above this statement is equivalent to saying that whenever  $f$  is a real-valued Lipschitz map on an open set  $G$  in  $X$  (with  $X^*$  separable) then any nonempty slice  $S$  of the set  $\Upsilon$  of Gâteaux derivatives

of  $f$  contains  $D_f(x)$  where  $x \in G$  and  $f$  is Fréchet differentiable at  $x$ . Recall that a slice  $S$  of  $\Upsilon$  is a set of the form  $S(\Upsilon, v, \delta)$  where  $v \in X$ ,  $\delta > 0$  and

$$S(\Upsilon, v, \delta) = \{T \in \Upsilon : Tv > \alpha - \delta, \alpha = \sup_{T \in \Upsilon} Tv\}.$$

A natural generalization of this assertion to maps into  $\mathbb{R}^n$  or more general spaces  $Y$  would be as follows: Let  $\Upsilon$  be again the set of Gâteaux derivatives of  $f$  and consider any slice  $S = S(\Upsilon, \{v_i\}_{i=1}^m, \{y_i^*\}_{i=1}^m, \delta)$  of  $\Upsilon$  where  $m \in \mathbb{N}$ ,  $\{v_i\}_{i=1}^m \subset X$ ,  $\{y_i^*\}_{i=1}^m \subset Y^*$ ,  $\delta > 0$  and

$$S(\Upsilon, \{v_i\}, \{y_i^*\}, \delta) = \{T \in \Upsilon : \sum_{i=1}^m y_i^*(Tv_i) > \alpha - \delta, \alpha = \sup_{T \in \Upsilon} \sum_{i=1}^m y_i^*(Tv_i)\}.$$

Then  $S$  contains a point of the form  $D_f(x)$  where  $f$  is Fréchet differentiable at  $x$ . Unfortunately, this generalization does not hold even for maps into  $\mathbb{R}^2$  as the following example shows.

*Example 5.1 ([13]).* Let  $1 < p < \infty$  and let  $n$  be an integer with  $n > p$ . Then there is a Lipschitz map  $f = (f_1, f_2, \dots, f_n)$  from  $\ell_p$  into  $\mathbb{R}^n$  such that whenever  $f$  is Fréchet differentiable at a point  $x$  then  $\sum_{i=1}^n D_{f_i}(x)e_i = 0$ , where  $\{e_j\}_{j=1}^\infty$  denote the unit basis in  $\ell_p$ , but  $f$  is Gâteaux differentiable at  $x = 0$  and we have that  $\sum_{i=1}^n D_{f_i}(0)e_i = 1$ .

The next theorem shows however, that in the sense of  $\Gamma$ -almost everywhere the mean value theorem for Fréchet derivatives holds also for maps into spaces of dimension greater than one.

**THEOREM 5.2.** *Suppose that  $f : G \rightarrow Y$  is a Lipschitz mapping which is Fréchet differentiable at  $\Gamma$ -almost every point of an open subset  $G$  of a Banach space  $X$ . Then, for every slice  $S$  of the set of Gâteaux derivatives of  $f$ , the set of points  $x$  at which  $f$  is Fréchet differentiable and  $D_f(x) \in S$  is not  $\Gamma$ -null.*

*Proof.* Let  $S = S(\Upsilon, v_1, \dots, v_n, y_1^*, \dots, y_n^*, \delta)$ , where  $\Upsilon$  is the set of Gâteaux derivatives of  $f$ . We can assume, without loss of generality, that  $\sum_{k=1}^n \|v_k\| = \sum_{k=1}^n \|y_k^*\| = 1$ . Let  $x_0 \in G$  be such that  $f$  is Gâteaux differentiable at  $x_0$  and  $D_f(x_0) \in S(\Upsilon, v_1, \dots, v_n, y_1^*, \dots, y_n^*, \delta/4)$ . Define  $f_0(y) = f(x_0) + D_f(x_0)(y - x_0)$  and find  $r > 0$  such that  $\|f(x_0 + v) - f_0(x_0 + v)\| \leq \delta\|v\|/8$  for  $v \in \text{span}\{v_k\}_{k=1}^n$  and  $\|v\| \leq r$ . Let  $\gamma_0(t) = x_0 + r \sum_{k=1}^n t_k v_k$  and consider any  $\gamma \in \Gamma(X)$  such that  $\|\gamma - \gamma_0\|_{\leq n} < \delta r/8(1 + \text{Lip}(f))$ . For any  $t \in T$  we have

$$\begin{aligned} \|f(\gamma(t)) - f_0(\gamma_0(t))\| &\leq \|f(\gamma(t)) - f(\gamma_0(t))\| + \|f(\gamma_0(t)) - f_0(\gamma_0(t))\| \\ &\leq \text{Lip}(f)\|\gamma - \gamma_0\|_0 + \delta\|\gamma_0(t) - x_0\|/8 < \delta r/4. \end{aligned}$$

Since  $\sum_{k=1}^n \|y_k^*\| = 1$ , integration with respect to the first  $n$  variables and the divergence theorem imply that

$$\int_T \sum_{k=1}^n \frac{\partial}{\partial t_k} y_k^*(f(\gamma(t)) - f_0(\gamma_0(t))) d\mu(t) < \delta r/2.$$

It follows that for  $t$  belonging to a set of positive measure,

$$\begin{aligned} \sum_{k=1}^n \frac{\partial}{\partial t_k} y_k^*(f(\gamma(t))) &> \sum_{k=1}^n \frac{\partial}{\partial t_k} y_k^*(f_0(\gamma_0(t))) - r\delta/2 \\ &= r \sum_{k=1}^n y_k^*(D_f(x_0)(v_k)) - r\delta/2. \end{aligned}$$

If, in addition,  $\gamma$  is such that  $f$  is Fréchet differentiable at  $\gamma(t)$  for almost every  $t$ , we conclude that for  $t$  belonging to a set of positive measure,

$$\begin{aligned} \sum_{k=1}^n y_k^*(D_f(\gamma(t))(v_k)) &= \frac{1}{r} \sum_{k=1}^n y_k^*(D_f(\gamma(t))(D_k \gamma_0)) \\ &\geq \frac{1}{r} \sum_{k=1}^n \left( \frac{\partial}{\partial t_k} y_k^*(f(\gamma(t))) - \text{Lip}(f) \|D_k \gamma(t) - D_k \gamma_0(t)\| \|y_k^*\| \right) \\ &> \sum_{k=1}^n y_k^*(D_f(x_0)(v_k)) - 3\delta/4 \end{aligned}$$

and thus  $D_f(\gamma(t)) \in S$  for  $t$  from a set of positive measure.  $\square$

*Remark.* The same proof shows that if we assume only that  $f$  is Gâteaux differentiable  $\Gamma$ -almost everywhere and that we are given a  $\Gamma$ -null set  $N$  then every slice of the set of Gâteaux derivatives contains some  $D_f(x)$  where  $x \notin N$  and  $f$  is Fréchet differentiable at  $x$ .

**COROLLARY 5.3.** *In  $\ell_p$ ,  $1 < p < \infty$ , there are porous sets which are not  $\Gamma$ -null, and thus also real-valued Lipschitz functions whose sets of non Fréchet differentiability are not  $\Gamma$ -null.*

*Proof.* This is an immediate consequence of Corollary 3.12, Example 5.1 and Theorem 5.2.  $\square$

*Remark.* If  $\pi$  is a projection of  $X$  onto its subspace  $Y$  then a set  $A \subset Y$  is  $\Gamma$ -null in  $Y$  if and only if  $\pi^{-1}(A)$  is  $\Gamma$ -null in  $X$ . This follows by observing that the map  $\varphi : \Gamma(X) \rightarrow \Gamma(Y)$  defined by  $\varphi(\gamma) = \pi \circ \gamma$  is onto, hence open (by the open mapping theorem), and hence it maps residual sets onto residual sets. It follows that the statement of Corollary 5.3 holds also for any Banach space which contains some  $\ell_p$ ,  $1 < p < \infty$ , as a complemented subspace.

## 6. Remarks and problems

We start by stating the following differentiability conjecture.

**CONJECTURE 1.** *Assume that  $X^*$  is separable,  $\{g_i\}_{i=1}^\infty$  are Lipschitz mappings from  $X$  to  $\mathbb{R}$  and  $f_i$  are Lipschitz mappings from  $X$  to  $Y_i$  where each  $Y_i$  has the RNP. Then there is a point  $x \in X$  at which all  $g_i$  are Fréchet differentiable and all  $f_i$  are Gâteaux differentiable.*

It is unknown for which spaces  $X$  this conjecture holds. In particular, it is unknown if it holds for  $X = \ell_2$ . It may hold even for every  $X$  with  $X^*$  separable. The results of Sections 3 and 4 show that it holds for  $c_0$ ,  $C(K)$  with  $K$  compact countable, the Tsirelson space and subspaces of  $c_0$ . These are the first and so far only examples of infinite-dimensional Banach spaces for which it is known that the conjecture holds.

The differentiability conjecture is connected to another (well-known) open problem. Assume that  $X$  is Lipschitz equivalent to  $Y$ ; i.e. there is a Lipschitz mapping  $f$  from  $X$  onto  $Y$  so that  $f^{-1}$  exists and is also Lipschitz. Must  $X$  be linearly isomorphic to  $Y$  if we assume that  $X$  is separable and has the RNP? It is trivial to check that if the Lipschitz equivalence is Fréchet differentiable at some point  $x_0$  then  $D_f(x_0)$  is a linear isomorphism from  $X$  onto  $Y$ . If  $f$  is only Gâteaux differentiable at  $x_0$  then in general  $D_f(x_0)$  will be only an isomorphism from  $X$  into  $Y$ . If however besides  $f$  being Gâteaux differentiable at  $x_0$  we also know that the functions  $g_i = y_i^* \circ f$ ,  $1 \leq i < \infty$  are Fréchet differentiable at  $x_0$ , where  $\{y_i^*\}$  is a norming sequence in the unit ball of  $Y^*$ , then  $D_f(x_0)$  must be surjective. All this is discussed in [3, Chap. 7]. Consequently,

**PROPOSITION 6.1.** *If  $X$  has the RNP and satisfies the differentiability conjecture, then every Banach space Lipschitz equivalent to  $X$  is linearly isomorphic to  $X$ . In particular, this holds if  $X$  is the Tsirelson space.*

In [3, Chap. 7] it is proved for a large class of spaces  $X$  that if  $Y$  is Lipschitz equivalent to  $X$  then  $Y$  is linearly isomorphic to  $X$ . The Tsirelson space is the first known example where the isomorphism can always be taken to be the Gâteaux derivative of the Lipschitz equivalence at a suitable point.

A similar situation occurs for Lipschitz quotient maps introduced and studied in [2].

**PROPOSITION 6.2.** *If  $X$  is separable and reflexive and satisfies the differentiability conjecture, then any Banach space  $Y$  which is a Lipschitz quotient of  $X$  is already a linear quotient of  $X$ . In particular, this holds for  $X$  the Tsirelson space.*

*Proof.* We prove that  $Y$  is reflexive (and thus has the RNP). Let  $f$  be a Lipschitz quotient map from  $X$  onto  $Y$ . It follows from [2, Theorem 3.18] that  $Y^*$  is separable. If  $\{y_i^*\}_{i=1}^\infty$  is a norm dense subset of  $Y^*$ , it follows from the differentiability conjecture that there is a point  $x_0 \in X$  at which all the functions  $y_i^* \circ f$  are Fréchet differentiable. This easily implies that  $y^* \circ f$  is Fréchet differentiable at  $x_0$  for every  $y^* \in Y^*$ . The mapping which assigns to every  $y^*$  the Fréchet derivative of  $y^* \circ f$  at  $x_0$  is an isomorphic embedding (since  $f$  is a Lipschitz quotient map). Hence  $Y^*$  is isomorphic to a subspace of  $X^*$  and thus  $Y$  is reflexive. Using again the differentiability conjecture, we get that for some  $x_1 \in X$  the Gâteaux derivative of  $f$  at  $x_1$  exists and all  $y^* \circ f$  ( $y^* \in Y^*$ ) are Fréchet differentiable at  $x_1$ . An argument similar to that of [2, Prop. 3.10] will now show that  $D_f(x_1)$  is a linear quotient map from  $X$  onto  $Y$ .  $\square$

Another related problem is the following:

*Problem 6.3.* Let  $f$  be a Lipschitz equivalence from a separable Banach space  $X$  with the RNP onto a Banach space  $Y$ . If  $A \subset X$  is  $\Gamma$ -null, does this imply that  $f(A)$  is  $\Gamma$ -null?

For Haar null or Gauss null sets the answer to the analogue of this question is negative even for  $C^\infty$  diffeomorphisms (see [3, Chap. 5] for details and references). For  $\Gamma$ -null sets the situation is considerably better; since for any  $C^1$  diffeomorphism of  $X$  onto  $Y$  the mapping  $\psi_f(\gamma) = f \circ \gamma$  is a homeomorphism of  $\Gamma(X)$  onto  $\Gamma(Y)$ , we have

**PROPOSITION 6.4.** *If  $f$  is a  $C^1$  diffeomorphism of a separable Banach space  $X$  onto a separable Banach space  $Y$  and  $A \subset X$  is  $\Gamma$ -null, then  $f(A) \subset Y$  is  $\Gamma$ -null.*

If the answer to Problem 6.3 were positive it would solve the problem whether Lipschitz equivalence implies isomorphism (for separable spaces with the RNP). Indeed, this would imply with the help of Theorem 2.5 that there is an  $x_0 \in X$  so that the Lipschitz equivalence has a Gâteaux derivative  $D_f(x_0)$  at  $x_0$  and that  $f^{-1}$  has a Gâteaux derivative at  $f(x_0)$ . This implies that  $D_f(x_0)$  is a surjective isomorphism.

One difficulty in studying questions like Problem 6.3 stems from the fact that  $\gamma \in \Gamma(X)$  does not necessarily imply that  $f \circ \gamma \in \Gamma(Y)$ . However, for spaces with the RNP this problem is easy to overcome since  $\Gamma$ -null sets can be equivalently defined in the following way. For  $\gamma : T \rightarrow X$  denote the Lipschitz constant of  $\gamma$  along the  $i$ -th coordinate by

$$\text{Lip}_i(\gamma) = \sup_{r \in \mathbb{R}, t, t+re_i \in T} \|\gamma(t+re_i) - \gamma(t)\|/|r|$$

where  $e_i$  is the  $i$ -th unit vector in  $\ell_\infty$ . Consider the Fréchet space of continuous mappings  $\gamma : T \rightarrow X$  for which  $\text{Lip}_i(\gamma) < \infty$  for all  $i$  equipped with the topology generated by the semi-norms  $\sup_{t \in T} \|\gamma(t)\|$  and  $\text{Lip}_k(\gamma)$ , and define  $\tilde{\Gamma}(X)$  as the closure of the set of those mappings which depend only on finitely many coordinates. Then a Borel set  $N \subset X$  is  $\Gamma$ -null if and only if  $\mu\{t : \gamma(t) \in N\} = 0$  for  $\gamma$  in a residual set of  $\tilde{\Gamma}(X)$ .

If  $f$  is a Lipschitz equivalence from  $X$  onto  $Y$  then the map  $\varphi_f : \tilde{\Gamma}(X) \rightarrow \tilde{\Gamma}(Y)$  defined by  $\varphi_f(\gamma) = f \circ \gamma$  is one-to-one and onto. The map  $\varphi_f$  is however highly discontinuous. For giving a positive answer to Problem 6.3 it would be enough to know that  $\varphi_f$  maps residual sets onto residual sets. This however is false as the following example due to M. Csörnyei shows: Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a Lipschitz equivalence such that  $g(0) = -1/4$ ,  $g'(r) = 1$  for  $1/4 < r < 3/4$  and  $g'(r) = 2$  for  $r < 1/4$  and  $r > 3/4$ . Let  $\gamma_0 \in \tilde{\Gamma}(\mathbb{R})$  be defined by  $\gamma_0(t) = t_1$  and let

$$U = \{\gamma \in \tilde{\Gamma}(\mathbb{R}) : \sup_{t \in T} |\gamma(t) - \gamma_0(t)| < 1/4, \text{Lip}_1(\gamma - \gamma_0) < 1/4\}.$$

Whenever  $\gamma \in U$ , we denote  $M_\gamma = (g \circ \gamma)^{-1}(1/4, 3/4)$ , note that

$$\int_{M_\gamma} D_1(g \circ \gamma) d\mu = 1/2$$

and that up to a set of  $\mu$  measure zero,

$$M_\gamma = \{t \in T : D_1(g \circ \gamma)(t) < 5/4\} = \{t \in T : D_1(g \circ \gamma)(t) < 3/2\}.$$

Clearly, for  $0 < \varepsilon < 1/4$  no  $\tilde{\gamma}$  sufficiently close to  $g \circ \gamma(t) + \varepsilon t_1$  can have these properties. Hence  $\varphi_g : \tilde{\Gamma}(\mathbb{R}) \rightarrow \tilde{\Gamma}(\mathbb{R})$  maps  $U$  onto a nowhere dense subset of  $\tilde{\Gamma}(\mathbb{R})$ . It follows that  $\varphi_{g^{-1}}$  maps the residual set  $\tilde{\Gamma}(\mathbb{R}) \setminus \varphi_g(U)$  into  $\tilde{\Gamma}(\mathbb{R}) \setminus U$  which is not residual.

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